A Reexamination on Bell's Theorem

Tianrong Tan

Department of Physics, Qingdao University, Qingdao, Shandong 266071, China, Ttr359@126.com

Abstract: It is proved that Bell's inequality can be traced back to a classical probabilistic formula that is invalid clearly, while the spin correlation formula in quantum mechanics to a formula that has confirmed both by facts and quantum mechanics. Two misunderstandings about Bell's theorem are pointed out: Firstly, the hidden variable theory Bell's used is a special one, but it is regarded as a general form of such theories. Secondly, there are two outcomes obtainable from Bell's hypotheses: one is compatible with quantum mechanics and the other leads to Bell's inequality. Unfortunately, the former is regarded as a property of local hidden variable theory, while the latter as self-evident. G. Lochak has revealed the first misunderstanding, and thereby he pointed out that Bell's inequality has nothing to do with locality, but he did not find the second one, so that he still analyzed this problem starting from hidden variable theory, which is actually irrelative to Bell's inequality. [The Journal of American Science. 2005;1(2):42-50].

Key words: Bell's inequality; locality; hidden variable theory; spin correlation formula; classical probability theory

Introduction

It is well known that in 1960s J. S. Bell[1] advanced Bell's theorem: "Any a local hidden variable theory is impossible to repeat the whole statistical predictions of quantum mechanics". Bell's work contains two parts, one is to prove that any a local hidden variable theory leads to Bell's inequality, the other is to prove that:

(a) Bell's inequality and the spin correlation formula in quantum mechanics cannot hold true simultaneously.

In this paper, the meaning of Bell's theorem will be reexamined.

1 The Promises in Wigner's Proof

Above all, let us remember Bohm's perfect experiment: A source emits electron pairs in the singlet state continuously. Considering a system, say S, consisting of a pair of electrons e and e' in the singlet, the e flies towards the right and the e' towards the left. Soon afterwards, the e enters into a Stern-Gerlach device G_a oriented in the direction $\boldsymbol{a}$, in which it obtains spin (projection) measurement value σ_a. At the same time, the e' enters into device G_b oriented in the n the direction $\boldsymbol{b}$, and obtains τ_b. Because that the σ_a and the τ_b can be measured simultaneously, for given $x, y \in \{1, -1\}$ the probability $Pr(\sigma_a = x,\ \tau_b = y)$ is definable, and the experiment facts show that $\tau_b = -\sigma_a$ if $\boldsymbol{a} = \boldsymbol{b}$, in other words, $\sigma_b = -\tau_b$.

Due to the thinking of the original proof for Bell's theorem is rather complicated, let us start from another proof.

In a famous paper, E. P. Wigner[2] has given a proof, which is deemed to be the most compact for Bell's theorem. As far as I know, Wigner's train of thought is as follows.

Firstly, quantum mechanics gives that:

For arbitrary given unit vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and the angle $\gamma = \angle(\boldsymbol{a}, \boldsymbol{b})$, we have:

$$Pr(\sigma_b = 1 \mid \sigma_a = 1) = Pr(\sigma_b = -1 \mid \sigma_a = -1) = \cos^2\frac{\gamma}{2},$$

$$Pr(\sigma_b = 1 \mid \sigma_a = -1) = Pr(\sigma_b = -1 \mid \sigma_a = 1) = \sin^2\frac{\gamma}{2}. \quad (1)$$

This is also an experimental fact, which we call "polarization law" hereafter.

Starting from this fact and the probability multiplication formula

$Pr(\sigma_a = x,\ \sigma_b = y)\ =\ Pr(\sigma_a = x)\cdot Pr(\sigma_b = y \mid \sigma_a = x)$,

in addition, by means of the formulae

$$Pr(\sigma_a = x,\ \tau_b = -y)\ =\ Pr(\sigma_a = x,\ \sigma_b = y) \quad (2)$$

and

$$Pr(\sigma_a = 1)\ =\ 1/2, \quad (3)$$

Wigner gave

$$Pr(\sigma_a = 1,\ \tau_b = 1)\ =\ \frac{1}{2}\sin^2\frac{\gamma}{2}. \quad (4)$$

Secondly, according to local hidden variable theory, Wigner introduced a joint probability

$Pr(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z;\ \tau_a = x',\ \tau_b = y',\ \tau_c = z')$,

and from this expression concluded that:

$$Pr(\sigma_a = x,\ \tau_b = y')\ =\ \sum_T Pr(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z;\ \tau_a = x',\ \tau_b = y',\ \tau_c = z'), \quad (5)$$

in which the T under the summation sign is abbreviated from that $y, z, x', z' \in \{1,\ -1\}$.

Thirdly, from Eq. (4) and Eq. (5), an invalid inequality, which we call Wigner's inequality, is derived, and thereby Bell's theorem is proved.

Now, let us examine this proof carefully.

By the fact $\tau_b = -\sigma_b$, only when $x' = -x$, $y' = -y$, $z' = -z$, the joint probability introduced by Wigner is nonzero. So, Eq. (5) can be rewritten as that:

$$Pr(\sigma_a = x,\ \tau_b = -y)\ =\ \sum_z Pr(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z;\ \tau_a = -x,\ \tau_b = -y,\ \tau_c = -z),$$

in which the z under the summation sign indicates that $z \in \{1,\ -1\}$.

Applying Eq. (2), the left side of this formula is equal to $Pr(\sigma_a = x,\ \sigma_b = y)$. Also, considering that the joint probability in the right side is actually equivalent to $Pr(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z)$, the above formula and thereby Eq. (5) becomes

$$Pr(\sigma_a = x,\ \sigma_b = y)\ =\ \sum_z Pr(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z). \quad (6)$$

Besides, Eq. (2) has also been used in the derivation of Eq. (4) from Eq. (1).

It is thus seen that by means of Eq. (2), on the one hand, Eq. (1) gives Eq. (4), on the other hand, Eq. (6) becomes Eq. (5), whereas, Wigner's inequality results from Eq. (4) and Eq. (5). As such, a question appears naturally: Whether or not it is possible to derived Wigner's inequality without the intermediary Eq. (2)? The answer is positive.

2 The Conclusion of Wigner's Proof

Now, we derive Wigner's inequality from Eq. (1) and Eq. (6) under the promises those Wigner used.

From Eq. (3) and the probability multiplication formula it is concluded that for arbitrary directions ***m***, ***n*** and $x, y \in \{1,\ -1\}$, we have

$$Pr(\sigma_n = x,\ \sigma_m = y) = \frac{1}{2}Pr(\sigma_m = y \mid \sigma_n = x).$$

Together with Eq. (1) we obtain that: if $\theta\ =\ \angle(\boldsymbol{n},\ \boldsymbol{m})$, then

$$Pr(\sigma_n = 1,\ \sigma_m = -1)\ =\ \frac{1}{2}\sin^2\frac{\theta}{2}.$$

Introducing directions ***a***, ***b***, ***c*** and

$\alpha\ =\ \angle(\boldsymbol{b},\ \boldsymbol{c}),\quad \beta\ =\ \angle(\boldsymbol{a},\ \boldsymbol{c}),\quad \gamma\ =\ \angle(\boldsymbol{a},\ \boldsymbol{b})$,

we have

$$Pr(\sigma_b = 1,\ \sigma_c = -1)\ =\ \frac{1}{2}\sin^2\frac{\alpha}{2};$$

$$Pr(\sigma_a = 1,\ \sigma_c = -1)\ =\ \frac{1}{2}\sin^2\frac{\beta}{2};$$

$$Pr(\sigma_a = 1,\ \sigma_b = -1)\ =\ \frac{1}{2}\sin^2\frac{\gamma}{2}. \quad (7)$$

Besides, applying classical probability formula, it is easy to obtain that:

$$Pr(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z)\ =\ Pr(\sigma_a = x,\ \sigma_c = z,\ \sigma_b = y)\ =\ Pr(\sigma_b = y,\ \sigma_c = z,\ \sigma_a = x). \quad (8)$$

Cocsidering that

$$Pr(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z) \geqslant 0,$$

abbreviating the $Pr(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z)$ to $F(x, y, z)$, from Eq. (6) and the (8) it is concluded that:

Arbitrary giving ***a***, ***b***, ***c*** and $x, y, z \in \{1, -1\}$, there exits function $F(x, y, z) \geqslant 0$, such that:

$$Pr(\sigma_a = x,\ \sigma_b = y)\ =\ \sum_z F(x, y, z);$$

$$Pr(\sigma_a = x,\ \sigma_c = z) = \sum_y F(x,y,z);$$

$$Pr(\sigma_b = y,\ \sigma_c = z) = \sum_x F(x,y,z).$$

We call this proposition after "Bell's promise" in the following; the reason is that Bell has applied it unconsciously instead that he has found it.

Bell's promise gives that:

$$Pr(\sigma_a = 1,\ \sigma_b = -1) = F(1,-1,1) + F(1,-1,-1);$$
$$Pr(\sigma_a = 1,\ \sigma_c = -1) = F(1,1,-1) + F(1,-1,-1);$$
$$Pr(\sigma_b = 1,\ \sigma_c = -1) = F(1,1,-1) + F(-1,1,-1).$$

Considering that $F(x,y,z) \geqslant 0$, from the above three formulae we get the following inequality:

$$Pr(\sigma_a = 1,\ \sigma_b = -1) + Pr(\sigma_b = 1,\ \sigma_c = -1) \geqslant Pr(\sigma_a = 1,\ \sigma_c = -1).$$

Substituting Eq. (7) into this inequality, we have

$$\sin^2\frac{\gamma}{2} + \sin^2\frac{\alpha}{2} \geqslant \sin^2\frac{\beta}{2}.$$

That is just the Wigner's inequality.

However, taking that $\gamma = \alpha = \beta/2 = \pi/3$, and thereby

$$\sin^2\frac{\gamma}{2} = \sin^2\frac{\alpha}{2} = 1/4, \quad \sin^2\frac{\beta}{2} = 3/4,$$

Wigner's inequality is violated, which is just the outcome Wigner obtained.

It is thus seen that in order to obtain Wigner's inequality, we need not to introduce the joint probability $Pr(\sigma_a = x,\ \tau_b = y)$; instead, Bell's promise is indispensable. So, under otherwise identical conditions, from Wigner's proof we ought to conclude that:

(b) Polarization law and Bell's promise cannot hold true simultaneously.

3 A revision for Wigner's Proof

Eq. (3) is used in the course proving the (b), but as we see in the following this formula is false.

Let us consider the following process, a beam, say R, which is polarized along the direction $\boldsymbol{n}$, passes through a Stern-Gerlach device G_a and splits into two sub-beams, one of them, say A, which is polarized along the $\boldsymbol{a}$, departs from the G_a. Taking the $\sigma_n = 1$ as the precondition for this process, the expression $Pr(\sigma_a = 1)$ is abbreviated from the $Pr(\sigma_a = 1 | \sigma_n = 1)$, which indicates the probability that a single electron in the R departs the G_a finally. Clearly, the value of the $Pr(\sigma_a = 1)$, therefore, is dependent on the $\boldsymbol{n}$. It is thus seen that Eq. (3) is not true.

Fortunately, Eq. (3) is not necessary for proving the (b). We can give a version of Wigner's proof without the use of Eq. (3) as follows.

From the probability multiplication law we can see that only given the directions $\boldsymbol{a}$ and $\boldsymbol{b}$, the value of the $Pr(\sigma_a = x,\ \sigma_b = y)$ is undetermined. But we can prove that

$$\sum_{xy} xy\, Pr(\sigma_a = x,\ \sigma_b = y) = \boldsymbol{a}\cdot\boldsymbol{b}. \qquad (9)$$

from the polarization law as follows:

Let

$$I \equiv Pr(\sigma_a = 1,\ \sigma_b = 1) - Pr(\sigma_a = 1,\ \sigma_b = -1);$$
$$J \equiv Pr(\sigma_a = -1,\ \sigma_b = -1) - Pr(\sigma_a = -1,\ \sigma_b = 1).$$

Then,

$$\sum_{xy} xy\, Pr(\sigma_a = x,\ \sigma_b = y) = I + J.$$

Besides, by means of the probability multiplication formula and Eq. (1), we have

$$I = Pr(\sigma_a = 1)(\cos^2\frac{\gamma}{2} - \sin^2\frac{\gamma}{2});$$

$$J = Pr(\sigma_a = -1)(\cos^2\frac{\gamma}{2} - \sin^2\frac{\gamma}{2}).$$

Besides, as everyone knows that

$$Pr(\sigma_n = 1) + Pr(\sigma_n = -1) = 1$$

and

$$\cos^2\frac{\gamma}{2} - \sin^2\frac{\gamma}{2} = \cos\gamma = \boldsymbol{a}\cdot\boldsymbol{b}.$$

The above formulae give Eq. (9), from which we see that it is possible to define

$$E(\boldsymbol{a},\boldsymbol{b}) \equiv \sum_{xy} xy\, Pr(\sigma_a = x,\ \sigma_b = y). \qquad (10)$$

and thereby Eq. (9) is written as:

$$E(\boldsymbol{a},\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b}. \qquad (11)$$

This is an outcome resulted from polarization law.

On the other hand, from Bell's promise it is easy to conclude that:

(c) For arbitrary $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $x,y,z \in \{1,-1\}$, there exists $F(x,y,z) \geqslant 0$, such that:

$$E(\boldsymbol{a},\boldsymbol{b}) = \sum_{xyz} xy\, F(x,y,z),$$

$$E(\boldsymbol{a}, \boldsymbol{c}) = \sum_{xyz} xz\,F(x,y,z),$$

$$E(\boldsymbol{b}, \boldsymbol{c}) = \sum_{xyz} yz\,F(x,y,z).$$

The preceding two formulae in the (c) give

$$E(\boldsymbol{a}, \boldsymbol{b}) - E(\boldsymbol{a}, \boldsymbol{c}) = \sum_{xyz} x(y-z)\,F(x,y,z).$$

Considering that $x(y-z)\,F(x,y,z) = 0$ if $y = z$, the above formula can be rewritten as

$$E(\boldsymbol{a}, \boldsymbol{b}) - E(\boldsymbol{a}, \boldsymbol{c}) = \sum_{y \neq z} x(y-z)\,F(x,y,z).$$

Taking the absolute values on both sides, considering that the absolute value of the sum cannot be larger than the sum of the absolute values, we have

$$|\,E(\boldsymbol{a}, \boldsymbol{b}) - E(\boldsymbol{a}, \boldsymbol{c})\,| \leqslant \sum_{y \neq z} |\,x(y-z)\,F(x,y,z)\,|.$$

Also, considering that $F(x,y,z) \geqslant 0$, and thereby

$$|\,x(y-z)\,F(x,y,z)\,| = |\,x(y-z)|\,F(x,y,z);$$

and that $|\,x(y-z)| = 2$ if $y \neq z$, we can rewrite the above formula by

$$|\,E(\boldsymbol{a}, \boldsymbol{b}) - E(\boldsymbol{a}, \boldsymbol{c})\,| \leqslant 2\sum_{y \neq z} F(x,y,z).$$

On the other hand, considering that

$$\sum_{xyz} F(x,y,z) = 1,$$

the third formula of the (c) gives that

$$1 - E(\boldsymbol{b}, \boldsymbol{c}) = \sum_{xyz} (1 - yz)\,F(x,y,z) = 2\sum_{y \neq z} F(x,y,z).$$

The above two formulae give

$$|\,E(\boldsymbol{a}, \boldsymbol{b}) - E(\boldsymbol{a}, \boldsymbol{c})\,| \leqslant 1 - E(\boldsymbol{b}, \boldsymbol{c}). \qquad (12)$$

This is an outcome resulting from Bell's promise.

Now, we prove the (b) by reduction to absurdity: If the (b) is false, then the polarization law and Bell's promise hold true simultaneously. Thus, Eq. (11), which can be derived from the polarization law, and Eq. (12), which is able to result from Bell's promise, also hold true simultaneously. Substituting Eq. (11) into Eq. (12), we have

$$|\boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{c}| \leqslant 1 - \boldsymbol{b}\cdot\boldsymbol{c}.$$

However, taking that $\boldsymbol{b} \perp \boldsymbol{c}$ and the $\boldsymbol{a}$ runs parallel with $\boldsymbol{b} - \boldsymbol{c}$, then the above inequality gives $\sqrt{2} \leqslant 1$. It is thus seen that the above inequality is violated and thereby the (b) holds true. So, we have retrieved the (b) without the application of Eq. (3).

The (b) means that:

Firstly, Since that in Wigner's proof, only the joint probability $Pr(\sigma_a = x,\ \tau_b = y)$ is involved with the "interaction at a distance" and thereby with locality, now this probability is unnecessary, so that Wigner's proof has nothing to do with locality.

Secondly, it has been proved that Bell's promise results from classical probability theory and the polarization law is beyond all doubt. As a result, the unique meaning of the (b) is it confirms again that classical probability theory is not always applicable for micro process.

The next problem is how to understand the (a).

4 Spin Correlation Function

The correlativity between two random quantities is measured by a quantity called "correlation", of which the definition is the difference between the mean value of the product of the very two quantities and the product of two mean values. Clearly, both the mean values of the random quantities x and y in the expression $Pr(\sigma_a = x,\ \tau_b = y)$ are zero. So, the correlation between x and y is the mean values of the product xy, namely, is

$$P(\boldsymbol{a}, \boldsymbol{b}) \equiv \sum_{xy} xy\,Pr(\sigma_a = x,\ \tau_b = y). \qquad (13)$$

This formula is the definition for the spin correlation function of pair electrons in singlet.

It is easy to see that if Eq. (2) is true, then

$$P(\boldsymbol{a}, \boldsymbol{b}) = -E(\boldsymbol{a}, \boldsymbol{b}). \qquad (14)$$

As a result, Eq. (12) is equivalent to Bell's inequality

$$|P(\boldsymbol{a}, \boldsymbol{b}) - P(\boldsymbol{a}, \boldsymbol{c})| \leqslant 1 + P(\boldsymbol{b}, \boldsymbol{c}). \qquad (15)$$

and Eq. (11) is equivalent to that

$$P(\boldsymbol{a}, \boldsymbol{b}) = -\boldsymbol{a}\cdot\boldsymbol{b}, \qquad (16)$$

which is the spin correlation formula in quantum mechanics. Therefore, the (a) is equivalent to the (b). It is a pity that we do not know if Eq. (2) is true.

In Eq. (2), the $Pr(\sigma_a = x,\ \sigma_b = y)$ is a joint probability of two quantities those cannot be measured simultaneously. Such a probability is inconsiderable in quantum mechanics, but it is generally believed that provided by means of the experimental fact $\tau_b = -\sigma_b$, substituting $\tau_b = y$ by $\sigma_b = -y$ in the expression $Pr(\sigma_a = x,\ \tau_b = y)$, Eq. (2) is obtained immediately, so that Eq. (2) holds true beyond doubt. However, this thinking is wrong. In the Eq. (2), two sides denote different processes, the right side only involves one electron while the left involves a pair of electrons; the right side is concerned in two events occurring successively, while the left occurring simultaneously. For such two expressions containing widely different meaning, the substituting operation above has no physics significance.

Fortunately, an alternative to Eq. (2) can be found by a more complex way. This formula is

$$Pr(\sigma_a = x,\ \tau_b = y) = Pr(\sigma_b = -y,\ \sigma_a = x), \quad (17)$$

which will be got through three steps as follows:

The first step, we prove that Eq. (17) is true if the following two events are equivalent each other:

(i) The spin state of the e before entering the G_a is $\sigma_b = -y$.

(ii) The spin measurement value of the e' in the G_b is $\tau_b = y$.

If the (i) is equivalent to the (ii), then the product event of the $\sigma_a = x$ and $\tau_b = y$ is equivalent to the product event of the following two events:

1. The spin state of the e departing from the G_a is $\sigma_a = x$;
2. The spin state of the e entering the G_a is $\sigma_b = -y$.

By definition, the probability of this product event is just the $Pr(\sigma_b = -y,\ \sigma_a = x)$. Due to the probabilities of equivalent events are equal to each other, Eq. (17) is obtained.

The second step, we prove as follows that if the spin state of the e before entering the G_a is $\sigma_b = -1$, then the spin measurement value of the e' in the G_b is $\tau_b = 1$:

Considering that in the system S, the e and the e' is in the singlet, if the spin state of the e before entering the G_a is $\sigma_b = -1$, then the spin state of the e' before entering the G_b is $\tau_b = 1$. Because that the magnetic field direction of the G_b is the ***b***, from Eq. (1) it is concluded that the spin measurement value of the e' in the G_b is $\tau_b = 1$.

The third step, we prove that if the spin measurement value of the e' in the G_b is $\tau_b = 1$, then the spin state of the e before entering the G_a is $\sigma_b = -1$ by reduction to absurdity, namely, assuming that the spin measurement value of the e' in the G_b is $\tau_b = 1$, but the spin state of the e before entering the G_a is not $\sigma_b = -1$, then a contradiction will be appear as follows:

From the fact that $\sigma_b = -\tau_b$, it is concluded that if the spin measurement value of the e' in the G_b is $\tau_b = 1$, then we have that:

(iii) Provided that the magnetic field direction of the G_a is the ***b***, then the spin measurement value of the e in the G_a is $\sigma_b = -1$.

On the other hand, if the spin state of the e before entering the G_a is not $\sigma_b = -1$, then there will be two occurrences: one is $\sigma_b = 1$, the other is neither $\sigma_b = 1$ nor $\sigma_b = -1$. So, under the condition that the magnetic field direction of the G_a is ***b***, in the former case, the spin measurement value of the e in the G_a is $\sigma_b = 1$, namely, it is determined but not $\sigma_b = -1$; in the latter case, the spin measurement value of the e in the G_a is undetermined. Namely, if the spin state of the e before entering the G_a is not $\sigma_b = -1$, then we have:

(iv) Even if that the magnetic field direction of the G_a is the ***b***, the spin measurement value of the e in the G_a either $\sigma_b = -1$ or undetermined.

This outcome is the antithesis of the (iii), so that the contradiction that the proof needs is found.

From the second step and the third step, we has proved that the (i) is equivalent to the (ii), if $y = -1$. Similarly, we can prove that the same proposition is also true if $y = 1$. Consequently, we has proved that the (i) is equivalent to the (ii). Together with the outcome of the first step, Eq. (17) is proved.

From Eq. (17) it is easy to get

$$\sum_{xy} xy\, Pr(\sigma_a = x,\ \tau_b = y) = -\sum_{xy} xy\, Pr(\sigma_a = x,\ \sigma_b = y),$$

and thereby Eq. (14) is obtained, and thereby it is confirmed that the (a) is equivalent to the (b). As a result, we have proved that the meaning of Bell's theorem is only to confirm once more that classical probability theory is not always applicable for micro process.

Besides, it is easy to see from the above formula and the definition Eq. (13) we can derive directly that

$$P(\boldsymbol{a},\boldsymbol{b}) = -\sum_{xy} xy\, Pr\,(\sigma_a = x,\ \sigma_b = y)\,. \quad (18)$$

This formula results from classical probability theory completely. As we see, by means of this formula, it is possible to obtain directly Bell's inequality Eq, (15) from Bell's promise, as well as obtain directly the spin correlation formula in quantum mechanics Eq, (16) from the polarization law.

Indeed, to obtain Eq, (16) the quantum mechanics method is simpler and more direct. But the above derivation confirms that:

1. Despite the fact that Eq, (18) results from classical probability theory completely, it is still compatible with quantum mechanics.

2. Though Eq, (18) is a promise of Bell's inequality, it is not the reason that Bell's inequality is wrong.

The above two conclusions are of great importance to clear the meaning of Bell's work.

5 Local Hidden Variable Theory

In Bell's original work, a kind of hidden variable theory is used. This theory is based on a set of axioms, and thereby we call it "axiom hidden variable theory" hereafter. So-called local hidden variable theory is the conjunction of this theory and locality. Applying local hidden variable theory on the system S, a pair of electrons e and e' in the singlet, several results is obtained as follows:

I. Attaching a set of hidden variables, denoted by λ, to the state function of the S, it is possible to give the single measurement value of the first electron spin projection along the direction $\boldsymbol{a}$, say σ_a and that of the second along the $\boldsymbol{b}$, say τ_b, namely, there exists functions:

$$\sigma_a = A(\boldsymbol{a},\lambda), \qquad \tau_b = B(\boldsymbol{b},\lambda). \quad (19)$$

II. In the collection, say Λ, of all hidden variables, it is possible to define distribution function $\rho(\lambda) \geqslant 0$, such that the normalized condition $\int_\Lambda \rho(\lambda)\,\mathrm{d}\lambda = 1$ is satisfied, and for arbitrary subset $\Gamma \subseteq \Lambda$, it is obtained that

$$Pr\,(\lambda \in \Gamma) = \int_\Gamma \rho(\lambda)\,\mathrm{d}\lambda.$$

III. The spin correlation function of the system S can be expressed by

$$P(\boldsymbol{a},\boldsymbol{b}) = \int_\Lambda A(\boldsymbol{a},\lambda)\, B(\boldsymbol{b},\lambda)\, \rho(\lambda)\,\mathrm{d}\lambda. \quad (20)$$

By the I, the fact $\tau_b = -\sigma_b$ is expressed as

$$B(\boldsymbol{b},\lambda) = -A(\boldsymbol{b},\lambda),$$

so that Eq. (20) is rewritten as that

$$P(\boldsymbol{a},\boldsymbol{b}) = -\int_\Lambda A(\boldsymbol{a},\lambda)\, A(\boldsymbol{b},\lambda)\, \rho(\lambda)\,\mathrm{d}\lambda. \quad (21)$$

For given $x,y \in \{1,-1\}$, defining

$$\Gamma(x,y) \equiv \{\lambda \in \Lambda \mid A(a,\lambda) = x,\ A(b,\lambda) = y\},$$

from the II it is concluded that

$$\int_{\Gamma(x,y)} \rho(\lambda)\,\mathrm{d}\lambda = Pr\,(\lambda \in \Gamma(x,y)) =$$

$$Pr\,(A(a,\lambda) = x, A(b,\lambda) = y) = Pr\,(\sigma_a = x,\ \sigma_b = y)$$

and

$$\int_{\Gamma(x,y)} A(a,\lambda)\, A(b,\lambda)\, \rho(\lambda)\,\mathrm{d}\lambda = xy$$

$$Pr\,(\sigma_a = x,\ \sigma_b = y).$$

By the properties of integrals, we have

$$\int_\Lambda A(\boldsymbol{a},\lambda)\, A(\boldsymbol{b},\lambda)\, \rho(\lambda)\,\mathrm{d}\lambda =$$

$$\sum_{xy} \int_{\Gamma(x,y)} A(\boldsymbol{a},\lambda)\, A(\boldsymbol{b},\lambda)\, \rho(\lambda)\,\mathrm{d}\lambda.$$

Together with Eq. (21), the above two formulae give Eq. (18).

Also, giving unit vector $\boldsymbol{c}$ and variable $z \in \{1,-1\}$ and another subset

$$\Gamma(x,y,z) \equiv$$

$$\{\lambda \in \Lambda \mid A(\boldsymbol{a},\lambda) = x,\ A(\boldsymbol{b},\lambda) = y,\ A(\boldsymbol{c},\lambda) = z\},$$

from the II it is concluded that:

$$Pr\,(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z) = \int_{\Gamma(x,y,z)} \rho(\lambda)\,\mathrm{d}\lambda.$$

Considering that

$$\int_{\Gamma(x,y)} \rho(\lambda)\,\mathrm{d}\lambda = \sum_z \int_{\Gamma(x,y,z)} \rho(\lambda)\,\mathrm{d}\lambda,$$

we obtain Eq. (6), namely,

$$Pr\,(\sigma_a = x,\ \sigma_b = y) = \sum_z Pr\,(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z)\,.$$

Similarly, from the I and the II it is also concluded that

$$Pr\,(\sigma_a = x,\ \sigma_c = z) = \sum_y Pr\,(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z)\,;$$

$$Pr\,(\sigma_b = y,\ \sigma_c = z) = \sum_x Pr\,(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z)\,.$$

According to that $\rho(\lambda) \geqslant 0$, we have

$$Pr\,(\sigma_a = x,\ \sigma_b = y,\ \sigma_c = z) \ = \int_{\Gamma(x,\,y,\,z)} \rho\,(\lambda)\,\mathrm{d}\lambda \ \geqslant\ 0.$$

The above formulae give Bell's promise.

As we know, Eq. (18) and Bell's promise give Bell's inequality. Therefore, Bell's inequality can result from I, II and III, namely, result from so-called local hidden variable theory. This is just the first outcome in Bell's work.

6 Bell's Misunderstandings

Since that the so-called local hidden variable theory is the conjunction of locality and axiom hidden variable theory, which is left intact classical probability theory, whereas, it is well known that classical probability theory is probably inapplicable for micro processes, it has nothing essential new that so-called local hidden variable theory is impossible to repeat the whole statistical predictions of quantum mechanics. Though it is hard to say that this "theorem" is wrong, we must point that this "theorem" is misunderstood by Bell.

Despite that Bell is entitled to define the conjunction of locality and axiom hidden variable theory as "local hidden variable theory", it is yet impossible from this definition to conclude that any a hidden variable theory must be left intact the classical probability theory, or go a step further, the realism, which is generally regarded as the philosophy foundation for hidden variable theory, requires restoring the probability calculation method of classical probability theory and excluding that of quantum mechanics in micro processes. Unfortunately, since that axiom hidden variable theory is regarded as the general form of hidden variable theory, this absurd reasoning is actually tacitly approved and it goes so far as to conclude that locality is in conflict with realism, in other words, so-called "local realism" has been rejected from the violation of Bell's inequality. Perhaps, as viewed from the future physicists, this is the most monstrous event in the physical history.

Except regarding axiom hidden variable theory as the general form of hidden variable theory, Bell has another misunderstanding for Bell's inequality. As we know, Eq. (18) is compatible with quantum mechanics. Even only starting from that Bell's inequality is incompatible with quantum mechanics, we can confirm what leads to Bell's inequality is Bell's promise instead of Eq. (18).

Unfortunately, in the Bell's work, the application of Bell's promise is indirect and unconscious, and the course to derive Bell's inequality is quite flexuous, Bell's inequality is actually imputed to Eq. (18), and thereby finally imputed to Eq. (19) and Eq. (20), instead of from the application for Bell's promise. As viewed from the practice, this understanding is still more mortal.

Just due to these two understandings, it seems that Bell's theorem give a criterion between quantum mechanics and "local realism", and thereby a continued fanaticism is started. Some persons try hard to find more unimaginable philosophy conclusions from the violation of Bell's inequality, while the others rack their brains for discovering some hidden hypotheses in the promises concerning the functions $A\,(\boldsymbol{a}, \lambda)$, $B\,(\boldsymbol{b}, \lambda)$ and $\rho\,(\lambda)$, so as to save locality and realism simultaneously.

7 Lochak's Objection

It seems that the first person who took objection against Bell's work is a France physicist G. Lochak[3,4]. In the [3], he said that:

"Bell's reasoning involves not only a hypothesis on the local character of the theory, but also a hypothesis, which consists in the admission that such a theory must restore the classical probabilistic pattern simultaneously in the statistics of all measurement results. But it leads immediately to a contradiction with the calculation of the mean values in wave mechanics since the latter violates the usual probabilistic pattern; therefore it is not astonishing to 'discover' afterward an incompatibility between the results of wave mechanics and those ascribed to hidden-variable theories."

The main argument Lochak advanced can be, so far as I understand, signified as follows: In the original derivation of Bell's inequality, the integral (20) is introduced. Such a formula can be certainly agreed with, but the problem is that: Is it possible to introduce the same $\rho\,(\lambda)$ for different mean values? For example, is it possible to obtain the formula

$$P(\boldsymbol{a}, \boldsymbol{b}) - P(\boldsymbol{a}, \boldsymbol{c}) = \int_{\Lambda} [A(\boldsymbol{a},\lambda)\, B(\boldsymbol{b},\lambda) - A(\boldsymbol{a},\lambda)\, B(\boldsymbol{c},\lambda)]\, \rho(\lambda)\, \mathrm{d}\lambda$$

from Eq. (20)? The Lochak's answer is negative.

By means of the relation

$$B(\boldsymbol{b},\lambda) = -A(\boldsymbol{b},\lambda),$$

Eq. (20) becomes Eq. (21), and thereby the question only involves one electron. For given subset $\Gamma(x,y)$, it is got that

$$\int_{\Gamma(x,y)} \rho(\lambda)\mathrm{d}\lambda = Pr(\sigma_a = x,\ \sigma_b = y).$$

Herein, the expression $Pr(\sigma_a = x,\ \sigma_b = y)$ is a joint probability! But, for quantum probabilities, because of uncertainty principle, such joint probability is indefinable. It is thus seen that in the derivation of Bell's inequality, a classical probability formula is used.

As a result, in the [4] Lochak wrote that:

"The experimental violation of Bell's inequalities has nothing to do, in my opinion, with the so-called 'non locality' or 'non separability'. This violation simply signifies that quantum probabilities are not classical probabilities! This is a fact which was discovered more than half a century ago, which it is nice to see confirmed again, but which perhaps does not need so many sophisticated and beautiful experiments in order to be accepted."

It seems that no one objected to Lochak's thesis as viewed from mathematics-physics, but this thesis does not yet rivet the attention of physicists clearly. It has caused the discussion about Bell's theorem to be meaningless, but up to now, this theorem is still a problem in great demand.

On the other hand, the Lochak's thesis is unsatisfactory yet. Lochak believes that provided it has been pointed out that classical probability theory, especially the joint probabilities, is used in the derivation for Bell's inequality, locality is divorced from Bell's inequality, but this argument is insufficient.

According to Lochak's viewpoint, the question resolves itself into the joint probability $Pr(\sigma_a = x,\ \sigma_b = y)$ and thereby into Eq. (18), but that is not the question!

As we know, classical probability theory is not always inapplicable for micro processes, so that merely from the fact that classical probability theory has been used in the derivation for Bell's inequality, we cannot conclude that Bell's inequality is wrong. It remains to clear which formula in classical probability theory leads to Bell's inequality and why this formula is inapplicable for micro processes?

Lochak upholds de Broglie's point that hidden variables must obey the classical probability theory laws. Starting from this point the translation from hidden variable relations into the measurement value relations is a complex course. It seems that such a hidden variable theory only makes things complicated and makes no contribution to clear the origin of Bell's inequality. As seen, Bell's inequality has actually nothing to do with hidden variables.

When enumerating the classical probability formulae, Lochak expressed the multiplication formula by

$$Pr(A \cdot B) = Pr(A) \cdot Pr(B|A) = Pr(B) \cdot Pr(A|B).$$

Herein, Lochak gave actually two formulae; one is

$$Pr(A \cdot B) = Pr(A) \cdot Pr(B|A); \qquad (22)$$

and the other is

$$Pr(A \cdot B) = Pr(B \cdot A). \qquad (23)$$

Within the bounds that the $Pr(B|A)$ is meaningful, Eq. (22) is derivable from the probability frequency definition (to derive the multiplication formula from probability frequency definition requires a slight revision for such definition), while Eq. (23) from the Boolean algebra for event calculation. It is generally believed that the former is certainly applicable for micro processes, while the latter is not. Lochak did not distinguished between these two hypotheses and thereby he did not find what leads really to Bell's inequality is Bell's promise and the reason why Bell's promise is inapplicable for micro processes is the unreasonable application of the Boolean algebra for event calculation.

Despite all this, the Lochak's work is still of great important. He is the first one who pointed out that Bell's inequality is irrelevant to locality and thereby to the so-called action at a distance. If this argument is accepted, people will lose interest for Bell's theorem immediately.

8 Conclusion

To sum up, it has been seen that Bell's inequality can be tracked back to Bell's promise, which is a formula in classical probability theory, while the spin correlation formula in quantum mechanics to the polarization law, which is established both by facts and quantum mechanics. Consequently, two outcomes are obtained.

Firstly, the mathematical content of Bell's theorem, namely, the spin correlation formula in quantum mechanics and Bell's inequality cannot hold true simultaneously, is very mediocre, of which the reasoning needs neither locality nor hidden variable theory.

Secondly, it is predictable that the spin correlation formula in quantum mechanics will be approved experimentally and Bell's inequality will be violated. As a result, the experiments about Bell's inequality in the 1970s give us nothing essential new.

This conclusion perhaps is disappointing. It is generally believed that Bell's theorem is "a progress of great importance in physics". Now, it is revealed that this theorem is actually a domestic shame that is not suitable for making public.

Correspondence to:

Tianrong Tan
Department of Physics
Qingdao University
Qingdao, Shandong 266071, China,
Ttr359@126.com

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