Comparison of Direct And Indirect Boundary Element Methods For The Calculation of Potential Flow Around An Elliptic Cylinder With Linear Element Approach

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Abstract: In this paper, a comparison of direct and indirect boundary element methods is applied for calculating the potential flow field (i.e. velocity distribution) around an elliptic cylinder with linear element (i.e. a new)approach. To check the accuracy of the method, the computed flow velocity is compared with the analytical solution for the flow over the boundary of an elliptic cylinder. [Journal of American Science 2010; 6(2):70-74]. (ISSN: 1545-1003).

Keywords: Boundary element methods, Potential flow, Velocity distribution, Elliptic cylinder., Linear element

1. Introduction

From the time of fluid flow modeling, it had been struggled to find the solution of a complicated system of partial differential equations (PDE) for the fluid flows which needed more efficient numerical methods. With the passage of time, many numerical techniques such as finite difference method, finite element method, finite volume method and boundary element method etc. came into beings which made possible the calculation of practical flows. Due to discovery of new algorithms and faster computers, these methods were evolved in all areas in the past. These methods are CPU time and storage hungry. One of the advantages is that with boundary elements one has to discretize the entire surface of the body, whereas with domain methods it is essential to discretize the entire region of the flow field. The most important characteristics of boundary element method are the much smaller system of equations and considerable reduction in data which is prerequisite to run a computer program efficiently. These method have been successfully applied in a number of fields, for example elasticity, potential theory, elastostatics and elastodynamics (Brebbia, 1978; Brebbia and Walker, 1980).Furthermore, this method is well–suited to problems with an infinite domain. From above discussion, it is concluded that boundary element method is a time saving, accurate and efficient numerical technique as compared to other numerical techniques which can be classified into direct boundary element method and indirect boundary element method. The direct method takes the form of a statement which provides the values of the unknown variables at any field point in terms of the complete set of all the boundary data. Whereas the indirect method utilizes a distribution of singularities over the boundary of the body and computes this distribution as the solution of integral equation. The direct boundary element method was used for flow field calculations around complicated bodies (Morino et al.,1975,Mushtaq, 2008 & 2009). While the indirect method has been used in the past for flow field calculations surrounding arbitrary bodies (Hess and Smith, 1967; Hess, 1973, Muhammad, 2008)

2. Velocity Distribution

Consider the flow past an elliptic cylinder of semi axes a and b with center at the origin and let the onset flow be the uniform stream with velocity U in the positive direction of the x-axis as shown in figure (1).

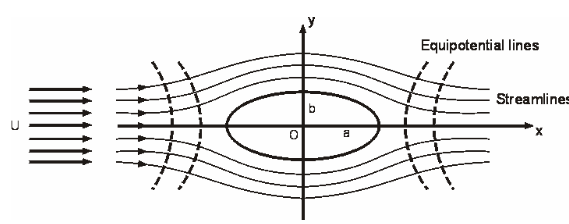

Figure 1: Flow past an elliptic cylinder

The magnitude of the exact velocity distribution over the boundary of the elliptic cylinder is given by (Milne-Thomson, 1968; Shah, 2008,Mushtaq,2009).

$$V = U(a+b)\frac{a\,y}{\sqrt{b^4x^2+a^4y^2}} \qquad (1)$$

Now the condition to be satisfied on the boundary of an elliptic cylinder is (Muhammad,2008;Mushtaq,2009)

$$\hat{n}\cdot\vec{V} = 0 \qquad (2)$$

where $\hat{n}$ is the unit normal vector to the boundary of the cylinder.

Since the motion is irrotational, $\vec{V} = -\nabla\Phi$

where Φ is the total velocity potential. Thus equation (2) becomes

$$\hat{n}\cdot(-\nabla\Phi) = 0$$

or $$\frac{\partial\Phi}{\partial n} = 0 \qquad (3)$$

Now the total velocity potential Φ is the sum of the perturbation velocity potential and the velocity potential of the uniform stream $\phi_{u.s}$

i.e. $$\Phi = \phi_{u.s} + \phi_{e.c} \qquad (4)$$

or $$\frac{\partial\Phi}{\partial n} = \frac{\partial\phi_{u.s}}{\partial n} + \frac{\partial\phi_{e.c}}{\partial n}$$

which on using equation (3) becomes

$$\frac{\partial\phi_{e.c}}{\partial n} = -\frac{\partial\phi_{u.s}}{\partial n} \qquad (5)$$

where $\phi_{e.c}$ is the velocity potential at the surface of the elliptic cylinder.

But the velocity potential of the uniform stream is given as

$$\phi_{u.s} = -U\,x$$

Then

$$\frac{\partial\phi_{u.s}}{\partial n} = -U\frac{\partial x}{\partial n} = -U\left(\hat{n}\cdot\hat{i}\right) \qquad (6)$$

Thus from (5) and (6)

$$\frac{\partial\phi_{e.c}}{\partial n} = U\left(\hat{n}\cdot\hat{i}\right) \qquad (7)$$

The equation of the boundary of the elliptic cylinder is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad (8)$$

Thus from (7)

$$\frac{\partial\phi_{e.c}}{\partial n} = U\frac{x\,b^2}{\sqrt{b^4x^2+a^4y^2}} \qquad (9)$$

Equation (9) is the boundary condition which must be satisfied over the boundary of an elliptic cylinder.

Now for the approximation of the boundary of the elliptic cylinder, the coordinates of the extreme points of the boundary elements can be generated within the computer program as follows:

Let the boundary of an elliptic cylinder is divided into linear elements. In this case the nodes where the boundary conditions are specified are at the intersection of the elements. The boundary of the cylinder can be divided into m elements in the clockwise direction by using the formula

$$\theta_k = [(m+2)-2k]\,\pi/m, \quad k = 1, 2, \ldots, m \qquad (10)$$

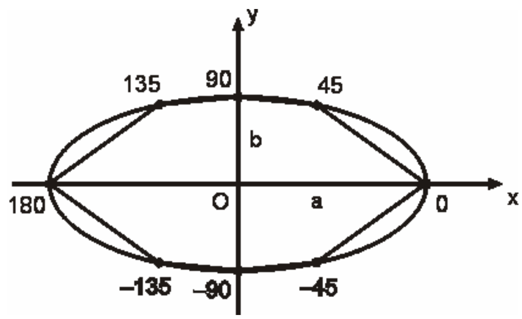

Figure 2. Discretization of the elliptic cylinder into 8 linear boundary elements

Then the coordinates of the extreme points of these m elements are calculated from

$$\left.\begin{aligned} x_k &= a\cos\theta_k \\ y_k &= b\sin\theta_k \end{aligned}\right\}, \quad k = 1, 2, \ldots, m \qquad (11)$$

Take $m = 8$, $a = 2$ and $b = 1$.

Thus the coordinates of the middle node of each boundary element are given by

$$\left.\begin{aligned} x_m &= (x_k + x_{k+1})/2 \\ y_m &= (y_k + y_{k+1})/2 \end{aligned}\right\} k, \quad m = 1, 2, \ldots, 8 \qquad (12)$$

and therefore the boundary condition (9) in this case takes the form

$$\frac{\partial\phi_{e.c}}{\partial n} = U\frac{x_m b^2}{\sqrt{b^4x_m^2+a^4y_m^2}}.$$

The velocity U of the uniform stream is also taken as unity.

The following table shows the comparison of the direct and indirect boundary element methods for analytical and computed velocities over the boundary of an elliptic cylinder for 8,16 and 32 linear boundary elements.

Table 1: The comparison of the computed velocity with exact velocity over the boundary of an elliptic cylinder using 8 linear boundary elements.

Element	x-Coordinate	y-Coordinate	$R=\sqrt{x^2+y^2}$	Computed Velocity Using DBEM	Computed Velocity Using IBEM	Analytical Velocity
1	-1.71	.35	.17433E+01	.87357E+00	.82676E+00	.95693E+00
2	-.71	.85	.11084E+01	.14583E+01	.14498E+01	.14688E+01
3	.71	.85	.11084E+01	.14583E+01	.14498E+01	.14688E+01
4	1.71	.35	.17433E+01	.87357E+00	.82676E+00	.95693E+00
5	1.71	-.35	.17433E+01	.87357E+00	.82676E+00	.95693E+00
6	.71	-.85	.11084E+01	.14583E+01	.14498E+01	.14688E+01
7	-.71	-.85	.11084E+01	.14583E+01	.14498E+01	.14688E+01
8	-1.71	-.35	.17433E+01	.87357E+00	.82676E+00	.95693E+00

Table 2: The comparison of the computed velocity with exact velocity over the boundary of an elliptic cylinder using 16 linear boundary elements.

Element	x-Coordinate	y-Coordinate	$R=\sqrt{x^2+y^2}$	Computed Velocity Using DBEM	Computed Velocity Using IBEM	Analytical Velocity
1	-1.92	.19	.19334E+01	.51847E+00	.51430E+00	.55447E+00
2	-1.63	.54	.17196E+01	.11710E+01	.11735E+01	.12010E+01
3	-1.09	.82	.13611E+01	.14180E+01	.14159E+01	.14227E+01
4	-.38	.96	.10353E+01	.14950E+01	.14908E+01	.14926E+01
5	.38	.96	.10353E+01	.14950E+01	.14908E+01	.14926E+01
6	1.09	.82	.13611E+01	.14180E+01	.14159E+01	.14227E+01
7	1.63	.54	.17196E+01	.11710E+01	.11735E+01	.12010E+01
8	1.92	.19	.19334E+01	.51848E+00	.51430E+00	.55447E+00
9	1.92	-.19	.19334E+01	.51847E+00	.51430E+00	.55447E+00
10	1.63	-.54	.17196E+01	.11710E+01	.11735E+01	.12010E+01
11	1.09	-.82	.13611E+01	.14180E+01	.14159E+01	.14227E+01
12	.38	-.96	.10353E+01	.14950E+01	.14908E+01	.14926E+01
13	-.38	-.96	.10353E+01	.14950E+01	.14908E+01	.14926E+01
14	-1.09	-.82	.13611E+01	.14180E+01	.14159E+01	.14227E+01
15	-1.63	-.54	.17196E+01	.11710E+01	.11735E+01	.12010E+01
16	-1.92	-.19	.19334E+01	.51847E+00	.51430E+00	.55447E+00

Table 3: The comparison of the computed velocity with exact velocity over the boundary of an elliptic cylinder using 32 linear boundary elements.

Element	x-Coordinate	y-Coordinate	$R=\sqrt{x^2+y^2}$	Computed Velocity Using DBEM	Computed Velocity Using IBEM	Analytical Velocity
1	-1.98	.10	.19832E+01	.28201E+00	.28299E+00	.28991E+00
2	-1.90	.29	.19264E+01	.76380E+00	.76608E+00	.77805E+00
3	-1.76	.47	.18170E+01	.10855E+01	.10871E+01	.10954E+01
4	-1.54	.63	.16631E+01	.12762E+01	.12766E+01	.12810E+01
5	-1.26	.77	.14786E+01	.13860E+01	.13855E+01	.13877E+01
6	-.94	.88	.12848E+01	.14490E+01	.14480E+01	.14491E+01
7	-.58	.95	.11139E+01	.14836E+01	.14824E+01	.14830E+01
8	-.20	.99	.10094E+01	.14990E+01	.14978E+01	.14982E+01
9	.20	.99	.10094E+01	.14990E+01	.14978E+01	.14982E+01
10	.58	.95	.11139E+01	.14836E+01	.14824E+01	.14830E+01
11	.94	.88	.12848E+01	.14490E+01	.14480E+01	.14491E+01
12	1.26	.77	.14786E+01	.13860E+01	.13855E+01	.13877E+01

13	1.54	.63	.16631E+01	.12762E+01	.12766E+01	.12810E+01
14	1.76	.47	.18170E+01	.10855E+01	.10871E+01	.10954E+01
15	1.90	.29	.19264E+01	.76380E+00	.76609E+00	.77805E+00
16	1.98	.10	.19832E+01	.28201E+00	.28298E+00	.28990E+00
17	1.98	-.10	.19832E+01	.28201E+00	.28299E+00	.28990E+00
18	1.90	-.29	.19264E+01	.76380E+00	.76608E+00	.77805E+00
19	1.76	-.47	.18170E+01	.10855E+01	.10871E+01	.10954E+01
20	1.54	-.63	.16631E+01	.12762E+01	.12766E+01	.12810E+01
21	1.26	-.77	.14786E+01	.13860E+01	.13855E+01	.13877E+01
22	.94	-.88	.12848E+01	.14490E+01	.14480E+01	.14491E+01
23	.58	-.95	.11139E+01	.14836E+01	.14824E+01	.14830E+01
24	.20	-.99	.10094E+01	.14990E+01	.14978E+01	.14982E+01
25	-.20	-.99	.10094E+01	.14990E+01	.14978E+01	.14982E+01
26	-.58	-.95	.11139E+01	.14836E+01	.14824E+01	.14830E+01
27	-.94	-.88	.12848E+01	.14490E+01	.14480E+01	.14491E+01
28	-1.26	-.77	.14786E+01	.13860E+01	.13855E+01	.13877E+01
29	-1.54	-.63	.16631E+01	.12762E+01	.12766E+01	.12810E+01
30	-1.76	-.47	.18170E+01	.10855E+01	.10871E+01	.10954E+01
31	-1.90	-.29	.19264E+01	.76381E+00	.76610E+00	.77805E+00
32	-1.98	-.10	.19832E+01	.28201E+00	.28297E+00	.28990E+00

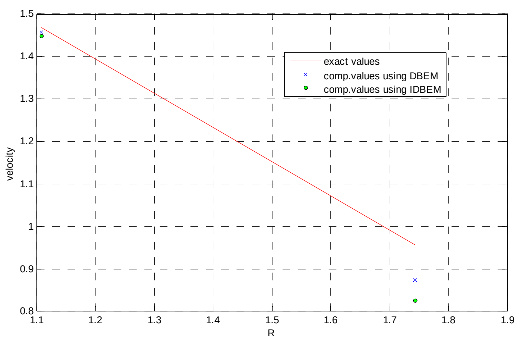

Figure 3. Comparison of computed and analytical velocity distributions over the boundary of an elliptic cylinder using 8 boundary elements with linear element approach.

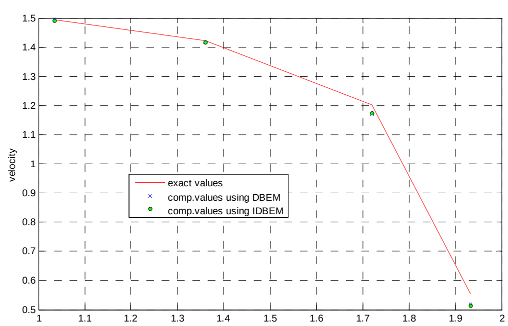

Figure 4. Comparison of computed and analytical velocity distributions over the boundary of an elliptic cylinder using 16 boundary elements with linear element approach.

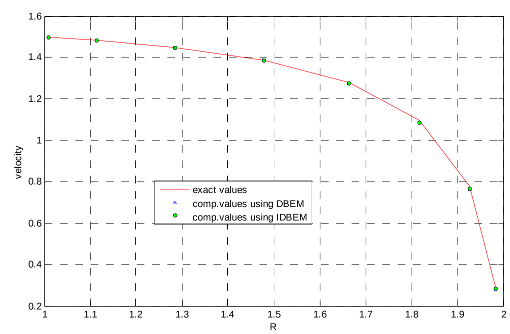

Figure 5. Comparison of computed and analytical velocity distributions over the boundary of an elliptic cylinder using 32 boundary elements with linear element approach.

3. Conclusion

A direct and indirect boundary element methods have been used for the calculation of potential flow around an elliptic cylinder with linear element(i.e. a new) approach. The calculated flow velocities obtained using these methods are compared with the analytical solutions for flows over the boundary of an elliptic cylinder . It is found that the results obtained overall with the indirect boundary element method for the flow field calculations are in excellent agreement with the analytical results for the body under consideration.

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References

[1]. Hess, J.L. and Smith, A.M.O.: "Calculation of potential flow about arbitrary bodies", Progress in Aeronautical Sciences, Pergamon Press 1967,8: 1-158.

[2]. Hess, J.L.: "Higher order numerical solutions of the integral equation for the two-dimensional Neumann problem", Computer Methods in Applied Mechanics and Engineering, 1973 :1-15.

[3]. Morino, L., Chen, Lee-Tzong and Suciu,E.O.: "A steady and oscillatory subsonic and supersonic aerodynamics around complex configuration", AIAA Journal,1975, 13(3): 368-374.

[4]. Milne-Thomson, L.M.: "Theoretical Hydrodynamics", 5th Edition, London Macmillan & Co. Ltd., 1968,158-161.

[5]. Brebbia, C.A.: "The Boundary element Method for Engineers", Pentech Press 1978.

[6]. Brebbia, C.A. and Walker, S.: " Boundary Element Techniques in Engineering" , Newnes-Butterworths 1980.

[7]. Shah, N.A. "Ideal Fluid Dynamics", A–One Publishers, Lahore–Pakistan 2008,420-436.

[8] Muhammad, G., Shah, N.A., & Mushtaq, M.: "Indirect Boundary Element Method for the Flow Past a Circular Cylinder with Linear Element Approach", International Journal of Applied Engineering Research 2008,3(12):1791-1798.

[9] Mushtaq, M., Shah, N.A. & Muhammad, G.: "Comparison of Direct and Indirect Boundary Element Methods for the Flow Past a Circular Cylinder with Linear Element Approach", Australian Journal of Basic and Applied Sciences Research, 2008, 2(4): 1052-1057.

[10] Mushtaq, M., Shah, N.A. & Muhammad, G.: "Comparison of Direct and Indirect Boundary Element Methods for the Calculation of Potential Flow Around an Elliptic Cylinder with Constant Element Approach", Australian Journal of Basic and Applied Sciences Research, 2009, 3(2): 1334-1339.

[11] Mushtaq, M., Shah, N.A. & Muhammad, G.: "Comparison of Direct and Indirect Boundary Element Methods for the Flow Past a Circular Cylinder with Constant Element Approach", Journal of American Science 2009: 5(4): 13–16.