The Difference Sequence Space Defined on Orlicz-Cesaro Function

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Abstract: The idea of difference sequence spaces was introduced by Kizmaz [4]. Recently, Subramanian [13] studied the difference sequence space $\ell_M(\Delta)$ defined on Orlicz function M. In this paper we introduce new sequence spaces that we call Orlicz-Cesaro difference sequence space and denote it by $Ces_M(\Delta)$, the difference paranormed sequence space $Ces_M(\Delta, p)$, and study some inclusion relations and completeness of this spaces. [Journal of American Science. 2010;6(10):25-30]. (ISSN: 1545-1003).

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Introduction

Orlicz [9] used the idea of Orlicz function to construct the space (L^M).

Lindentrauss and Tzafriri [5] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (1 \le p < \infty)$.

Subsequently different classes of sequence spaces defined by Parashar and Ghoudhary [10], Murasaleen et al. [6] Bekats and Altin [1], Tripathy et al. [14], Rao and Subramanian [2] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref [3].

Recall ([3],[9]) an Orlicz function is a function $M:[0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0)=0$,

$M(x)>0$ for $x>0$ and

$M(x) \to \infty$ as $x \to \infty$.

If convexity of Orlicz function M is replaced by $M(x+y) \le M(x)+M(y)$ then this function is called modulus function, introduced by Nakano [8] and further discussed by Ruckle [12] and Maddox [7]. By ω, we denote the space of all real or complex sequences. The sets of natural numbers and real numbers will denoted by $\mathbb{N}=\{1,2,3, \dots\}$, $\mathbb{R}$ respectively.

Lindentrauss and Tzafriri [5] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}$$

. The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

becomes a Banach space which is called Orlicz sequence space. For $M(t)=t^p$, $1 \le p < \infty$, the space ℓ_M coincide with the classical sequence space ℓ_p. A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a sub additive function $g: X \to \mathbb{R}$ such that $g(\theta)=0$, $g(-x)=g(x)$ and for any sequence (x_n) in X such that $g(x_n - x) \xrightarrow{n-\infty} 0$, and any sequence (α_n) in $\mathbb{R}$ such that $|\alpha_n - \alpha| \xrightarrow{n-\infty} 0$, we get $g(\alpha_n x_n - \alpha x) \xrightarrow{n-\infty} 0$.

The idea of difference sequence was first introduced by Kizmaz [4] write

$\Delta x_k = x_k - x_{k+1}$ for k=1,2,3, …. Let ω denote the set of all real or complex sequences, $\Delta : \omega \to \omega$ be the difference defined by $\Delta x = (\Delta x_k)_{k=1}^{\infty}$, and $M : [0,\infty) \to [0,\infty)$ be an Orlicz function; or a modulus function.

Let ℓ be the sequence of absolutely convergent series. Define a sequence space

$$\ell(\Delta) = \{x = (x_k) : \Delta x \in \ell\}.$$

The sequence space

$$\ell_M(\Delta) = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|\Delta x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}$$

With the norm

$$\| x \| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\Delta x_k|}{\rho}\right) \le 1 \right\},$$

becomes a Banach space which is called an Orlicz difference sequence space $\ell_M(\Delta)$, see [13].

The Cesaro-Orlicz sequence space Ces_M generated by Orlicz function M is defined by

$$Ces_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{\frac{1}{k}\sum_{i=1}^{k}|x_i|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}$$

And Ces_M with the norm

$$\| x \| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\frac{1}{k}\sum_{i=1}^{k}|x_i|}{\rho}\right) \le 1 \right\}$$

Is a Banach space (see [11]).We define the following sequence space

Definition:

$$Ces_M(\Delta) = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{\frac{1}{k}\sum_{i=1}^{k}|\Delta x_i|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}$$

With the norm

$$\| x \| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\frac{1}{k}\sum_{i=1}^{k}|\Delta x_i|}{\rho}\right) \le 1 \right\}.$$

Theorem(1):

$Ces_M(\Delta)$ Is a Banach space with the norm

$$\| x \| = \inf\left\{ \rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right) \le 1 \right\}.$$

Proof:

Let $x^{(i)}$ be any Cauchy sequence in $Ces_M(\Delta)$, where

$$x^{(i)} = (x_1^{(i)}, x_2^{(i)}, ...) \in Ces_M(\Delta) \; \forall\, i \in \mathbb{N}.$$

Let $r, x_0 > 0$ be fixed. Then for each $\frac{\varepsilon}{r x_0} > 0$ there exist a positive integer N such that $\| x^{(i)} - x^{(j)} \|_{\Delta} < \frac{\varepsilon}{r x_0} \; \forall i, j \ge N$ using the

definition of norm we get

$$\sum_{n=1}^{\infty} M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k^i - \Delta x_k^j|}{\| x^{(i)} - x^{(j)} \|_{\Delta}}\right) \le 1,$$

$\forall i, j \ge N$ then,

$$M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k^{(i)}-\Delta x_k^{(j)}|}{\|x^{(i)}-x^{(j)}\|_\Delta}\right)\le 1\ \forall n\ge 1$$ and. Hence we can find r>0 with $M\left(\frac{rx_0}{n}\right)>1$, such that

$$M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k^{(i)}-\Delta x_k^{(j)}|}{\|x^{(i)}-x^{(j)}\|_\Delta}\right)\le M\left(\frac{rx_0}{n}\right).$$

Since M is non-decreasing, this implies that

$$\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k^{(i)}-\Delta x_k^{(j)}|}{\|\Delta x^{i}-\Delta x^{j}\|_\Delta}\le\frac{rx_0}{n}$$

$$\Rightarrow\sum_{k=1}^{n}|\Delta x_k^{(i)}-\Delta x_k^{(j)}|\le rx_0\|x^{(i)}-x^{(j)}\|_\Delta<rx_0.\frac{\varepsilon}{rx_0}=\varepsilon$$

Since

$$\sum_{k=1}^{n}|\Delta x_k^{(i)}-\Delta x_k^{(j)}|\ge|\Delta x_k^{(i)}-\Delta x_k^{(j)}|\ for\ all\ 1\le k\le n$$

, we get

$|\Delta x_k^{(i)}-\Delta x_k^{(j)}|<\varepsilon \qquad \forall i,j\ge N$ .Therefore $\left(\Delta x_k^{(j)}\right)_{j=1}^{n}$ be a Cauchy Sequence in $\mathbb{R}$ (complete)

Then $\Delta x^{(j)}\to\Delta x$ as j→∞.Using the continuity of M We can find that

$$\sum_{n=1}^{N}M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k^{(i)}-\lim_{i\to\infty}\Delta x_k^{(j)}|}{\rho}\right)\le 1,$$ thus

$$\sum_{k=1}^{N}M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k^{(i)}-\Delta x_k|}{\rho}\right)\le 1.$$

Taking infimum of such ρ 's we get

$$\inf\left\{\rho>0:\sum_{n=1}^{N}M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k^{(i)}-\Delta x_k|}{\rho}\right)\le 1\right\}<\varepsilon,$$

for all $i\ge N$. Since $x^{(i)}\in Ces_M(\Delta)$ and M is continuous then $\Delta x^{(i)}\xrightarrow{i\to\infty}\Delta x\in Ces_M(\Delta)$, this completes the proof.

Theorem(2): $Ces_M\subseteq Ces_M(\Delta), M$ is a modulus function.

Proof: Let $x\in Ces_M$, then

$$\sum_{n=1}^{\infty}M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)<\infty$$, for some $\rho>0$, since

$\Delta x_k=x_k-x_{k+1}$

$$\Rightarrow\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\le\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}+\frac{\frac{1}{n}\sum_{k=1}^{n}|x_{k+1}|}{\rho}.$$

Since M is non-decreasing and modulus function

$$\Rightarrow\sum_{n=1}^{\infty}M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\le M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_k|}{\rho}\right)+M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|x_{k+1}|}{\rho}\right)<\infty$$

$\Rightarrow x\in Ces_M(\Delta)$.

Paranormed sequence spaces:

Let $p=(p_n)$ be any sequence of positive real numbers. Then in the same way we can also define the following sequence space

$$Ces_M(\Delta,p)=\left\{x\in\omega:\sum_{n=1}^{\infty}\left(M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right)^{p_n}<\infty,\exists\rho>0\right\}$$

.

Note: If $p_n = p$, for all $n \in \mathbb{N}$, then

$Ces_M(\Delta, p) = Ces_M(\Delta)$.

Theorem(3): $Ces_M(\Delta, p)$ is a complete paranormed space with

$$g^*(x) = \inf\left\{ \rho^{\frac{P_n}{H}} : \left[\sum_{n=1}^{\infty} \left(M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\Delta x_k|}{\rho} \right) \right)^{P_n} \right]^{\frac{1}{H}} \leq 1 \right\}, \text{ for}$$

$1 \leq p_n < \infty \quad \forall n \in \mathbb{N}$ and

$H = \max\{1, \sup_n p_n\}$.

Proof: Let $x^{(n)}$ be any Cauchy sequence in $Ces_M(\Delta, p)$, where $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)} ..) \ \forall i \in \mathbb{N}$. Let $r, x_o > 0$ be fixed. Then for each $\frac{\varepsilon}{rx_0} > 0$ there exist a positive integer N such that

$$g^*(x^{(i)} - x^{(j)}) < \frac{\varepsilon}{rx_0} \ \forall i, j \geq N,$$

Using the definition of paranom we get

$$\left[\sum_{n=1}^{\infty} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\Delta x_k^{(i)} - \Delta x_k^{(j)}|}{g^*(x^{(i)} - x^{(j)})} \right) \right]^{P_n} \right]^{\frac{1}{H}} \leq 1.$$

Since $1 \leq p_k < \infty$, it follows that

$$M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\Delta x_k^{(i)} - \Delta x_k^{(j)}|}{g^*(x^{(i)} - x^{(j)})} \right) \leq 1, \quad \forall i, j \geq N \quad \text{and}$$

$n \geq 1$. Hence we can find r>0 with $M(\frac{rx_0}{n}) > 1$ Such that

$$M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\Delta x_k^{(i)} - \Delta x_k^{(j)}|}{g^*(x^{(i)} - x^{(j)})} \right) \leq M(\frac{rx_0}{n}),$$

since M is non-decreasing

We get

$$\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\Delta x_k^{(i)} - \Delta x_k^{(j)}|}{g^*(x^{(i)} - x^{(j)})} \right) \leq \frac{rx_0}{n}$$

$$\Rightarrow \sum_{n=1}^{\infty} |\Delta x_k^{(i)} - \Delta x_k^{(j)}| \leq rx_0 g^*(x^{(i)} - x^{(j)}) < rx_0 \frac{\varepsilon}{rx_0} = \varepsilon$$

$$\Rightarrow |\Delta x_k^{(i)} - \Delta x_k^{(j)}| < \in$$

$\forall i, j \geq N$. Therefore $(\Delta x_k^{(j)})_{j=1}^{n}$ is a Cauchy sequence in $\mathbb{R}$(complete), then $\Delta x_k^j \xrightarrow{j \to \infty} \Delta x$, since M is continuous we can find that

$$\left[\sum_{n=1}^{N} \left[M\left(\frac{\frac{1}{n}\sum |\Delta x_k^{(i)} - Lim_{i \to \infty} \Delta x_k^{(j)}|}{\rho} \right) \right]^{P_n} \right]^{\frac{1}{H}} \leq 1$$

, thus

$$\left[\sum_{n=1}^{N} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\Delta x_k^{(i)} - \Delta x_k|}{\rho} \right) \right]^{P_n} \right]^{\frac{1}{H}} \leq 1,$$

taking infimum of such $\rho's$ we get

$$\inf\left\{ \rho^{\frac{P_n}{H}} : \left[\sum_{k=1}^{N} \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n} |\Delta x_k^{(i)} - \Delta x_k^{(i)}|}{\rho} \right) \right]^{P_n} \right]^{\frac{1}{H}} \leq 1 \right\} < \varepsilon$$

, $\forall i \geq N$, since $x^{(i)} \in Ces_M(\Delta, p)$ and M is continuous it follows that $x \in Ces_M(\Delta, p)$, then the proof is complete.

Theorem(4): Let $0 < p_n < q_n < \infty \ \forall n \in \mathbb{N}$,
Then $Ces_M(\Delta, p) \subseteq Ces_M(\Delta, q)$

Proof: Let $x \in Ces_M(\Delta, p)$

then $\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{P_n} < \infty$,

For some $\rho > 0$, we get, $M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right) \le 1$ for sufficiently large n, since M is non-decreasing. Hence we get

$$\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{q_n} \le \sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{P_n} < \infty$$

, thus $\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{q_n} < \infty$

$\Rightarrow x \in Ces_M(\Delta, q)$.This completes the proof.

Theorem(5):

(a) Let $0 < \inf\ p_n \le p_n \le 1 \ \forall n \in \mathbb{N}$. Then $Ces_M(\Delta, p) \subseteq Ces_M(\Delta)$.

(b) Let $1 \le p_n \le \sup_n\ p_n < \infty \ \forall n \in \mathbb{N}$. Then $Ces_M(\Delta) \subseteq Ces_M(\Delta, p)$.

Proof:(a) Let $x \in Ces_M(\Delta, p)$

$$\Rightarrow \sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{p_n} < \infty$$, we get $M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right) \le 1$,

For sufficiently large n, since $0 < \inf\ P_n \le 1 \ \forall n \in \mathbb{N}$

$$\Rightarrow \sum_{n=1}^{\infty} M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right) \le \sum_{n=1}^{\infty}\left(M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right)^{P_n} < \infty$$

, thus $x \in Ces_M(\Delta)$.

(b) Let $p_n \ge 1$ for each $n \in \mathbb{N}$ and sup $p_n < \infty$,
Let $x \in Ces_M(\Delta) \Rightarrow$

$\sum_{n=1}^{\infty} M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right) < \infty$, for some $\rho > 0$, for sufficiently large n we can get,

$$M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right) \le 1,$$

since $1 \le p_n \le \sup p_n < \infty$, we have that

$$\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{p_n} \le \sum_{n=1}^{n} M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right) < \infty$$

, thus $x \in Ces_M(\Delta, p)$.

Theorem (6): Let $0 \le p_n \le q_n$ and $\left(\frac{q_n}{p_n}\right)$ be bounded, then $Ces_M(\Delta, q) \subseteq Ces_M(\Delta, p)$.

Proof: Let $x \in Ces_M(\Delta, q)$

(i.e.) $\sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{q_n} < \infty$ .Let

$t_n = \left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{q_n}$ and $\lambda_n = \frac{q_n}{p_n}$ .Since

$p_n \le q_n$ therefore $0 \le \lambda_n \le 1$.Take0< $\lambda < \lambda_n$,
define $u_n = t_n(t_n \ge 1)$;
$u_n = 0(t_n < 1)$ and $v_n = 0(t_n \ge 1)$;
$u_n = t_n(t_n < 1)$. $t_n = u_n + v_n$. (i.e.)
$t_n^{\lambda_n} = u_n^{\lambda_n} + v_n^{\lambda_n}$ .Now it follows that
$u_n^{\lambda_n} \le u_n \le t_n$ and $v_n^{\lambda_n} \le v_n^{\lambda}$ (1).

$$\Rightarrow \sum_{n=1}^{\infty} t_n^{\lambda_n} = \sum_{n=1}^{\infty} (u_n + v_n)^{\lambda_n}.$$

$$\Rightarrow \sum_{n=1}^{\infty} t_n^{\lambda_n} \le \sum_{n=1}^{\infty} u_n^{\lambda_n} + \sum_{n=1}^{\infty} v_n^{\lambda_n}.$$

$$\Rightarrow \sum_{n=1}^{\infty} t_n^{\lambda_n} \le \sum_{n=1}^{\infty} t_n + \sum_{n=1}^{\infty} v_n^{\lambda}$$, by using equation (1), we get

$$\Rightarrow \sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{\lambda_n q_n} \le \sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{q_n}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{p_n} \le \sum_{n=1}^{\infty}\left[M\left(\frac{\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k|}{\rho}\right)\right]^{q_n}$$

.Then $Ces_M(\Delta, q) \subseteq Ces_M(\Delta, p)$.

Theorem(7): $Ces_M(\Delta, p)$ is a linear set over the set of complex numbers.

Proof: is easy so omitted.

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