Global Analysis of an Epidemic Model with General Incidence Rate.

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Abstract: A general four-dimensional SIQR epidemic model is considered. Threshold, equilibria and their stability are established. The dynamics of the system is discussed in the case of this general form of incidence rate. The global stability of both free-deisease and endemic equilibria are deduced. Hopf bifurcation , boundedness, dissipativity and persistence are studied.

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1. Introduction:

The technique of introducing quarantine in standard SIS and SIR epidemic models has received great interest in the last two decades (see for example [1], [5], [6], [7], and [21]). Over the centuries quarantine, which means forced isolation or stoppage of interactions with others, succeeded to reduce the transmission of human and animal diseases. In their recent paper Feng and Thieme [5] considered an $SIQR$ model with a quarantine class and showed that the qurantine can lead to periodic solutions. They considered in [6] a more general endemic model that includes $SEIQR$ models with arbitrary distributed periods of infection including quarantine. They proved extinction and persistence results. In [12], Hethcote et al discussed six endemic models with quarantined class. In this paper we consider an $SIQR$ model with an incidence term more general than those used by [3], [10] , [11], [12], [14], and [16]. Following [12],we assume that the total host population is partioned into susceptable, infectious, quarantine and recovered which densities denoted respectively by $S(t), I(t), Q(t)$ and $R(t)$. The natural birth rate denoted by A. Assume that infectious confers permanent immunity, so that, individuals can move from the I and Q classes to the R class, where $R(t)$ is the number of individuals with permanent immunity and $N(t) = S(t) + I(t) + Q(t) + R(t)$. In this paper we assume that the general incidence rate term $H(I,S)$ is differentiable, with $\frac{\partial H}{\partial I}$ and $\frac{\partial H}{\partial S}$ are nonnegative and finite for all I and S. More precisely we consider the 4-dimensional system of differential equations,

$$S' = A - \beta I H(I,S) - dS,$$

$$I' = \beta I H(I,S) - (\gamma + \delta + d + \alpha_1) I,$$

$$Q' = \delta I - (d + \alpha_2 + \varepsilon) Q,$$

(1.1) (1.1))

$$R' = \gamma I + \varepsilon Q - dR,$$

where A and d are positive constants and $\gamma, \delta, \varepsilon, \alpha_1$ and α_2 are non-negative constants. The constant A is the recruiment rate of susceptable corresponding to birth and immigratrion, β is the average number of adequate contact, d is the per capita natural mortality rate, δ is the rate constant for individuals leaving the infective compartment I for the quarantine compartment Q, γ and ε are the rates at which individuals recover and return to susceptable compartment S from compartment I and Q, respectively, and α_1 and α_2 represent the extra disease-related death rate constants in classes I and Q, respectively. The total population size $N(t)$ satisfies $N'(t) = A - dN - \alpha_1 I - \alpha_2 Q$, so that the population size $N(t)$ approached the carrying capacity $\frac{A}{d}$ when there is no disease. The differential equation for N implies that the solutions of (1.1) starting in the positive orthant R_4^+ either approach, enter, or remain in the subset

$$D = \left\{ (S,I,Q,R) : S \ge 0, I \ge 0, Q \ge 0, R \ge 0, S+I+Q+R \le \frac{A}{d} \right\}.$$

The model (1.1) is more general epidemiological model than those discussed in ([11],[14],[16], and [17]). It is known (see [11]) that the system (1.1) always has the disease-free equilibrium $P_0 = (\frac{A}{d}, 0, 0, 0)$. We define the quarantine reproduction number in the form $R_q = \dfrac{\beta(\frac{A}{d})}{(\gamma+\delta+d+\alpha_1)}$ where if $R_q > 1$, the region D contains also the endemic equilibrium $P^* = (S^*, I^*, Q^*, R^*)$, where

$$S^* = \frac{A}{d} - \frac{(\gamma+\delta+d+\alpha_1)}{d} I^*, Q^* = \frac{\delta I^*}{d+\alpha_2+\varepsilon}, \quad (1.2)$$

$$H(I^*, \frac{A}{d} - \frac{(\gamma+\delta+d+\alpha_1)}{d} I^*) = \frac{(\gamma+\delta+d+\alpha_1)}{\beta}.$$

The aim of this paper is to study the dynamic of (1.1) by different techniques with a generalized incidence term. We show that some of our obtained results may be applied for many special forms of $H(I,S)$. The organization of this paper is as follows. In section 2, we discuss the stability properties of the reduced 3-dimensional SIQ epidemic model. In section 3, we study the boundedness, dissipativity, persistence, global stability and Hopf bifurcation of solutions of the 4-dimensional model (1.1). Our technique in this section depends on [20]. The paper ends with numerical justifications and brief discussion in section 4.

2. A reduced SIQ epidemic model

Since the last equation in (1.1) is independent of the other equations (see[10], [11], [12] and [13]), we may start by discussing the 3-dimensional system,

$$S' = A - \beta I H(I,S) - -dS,$$

$$I' = \beta I H(I,S) - (\gamma+\delta+d+\alpha_1) I,$$

$$Q' = \delta I - (d+\alpha_2+\varepsilon) Q.$$

The dynamic of (1.1) in D is equivalent to that of (2.1) in the feasible region

$$\Gamma = \{(S,I,Q) \in R_+^3 : S+I+Q \le \frac{A}{d}\},$$

(2.2)

which can shown to be closed and positively invariant set with respect to (2.1) (see [14]). Letting $\partial\Gamma$ denotes the boundary of Γ and Γ^0 its interior , the system (2.1) always has disease-free equilibrium point $P_\circ = (\frac{A}{d},0,0) \in \partial\Gamma$.

Lemma 2.1. If $R_q \le 1$, then there exists a unique equilibrium point $P_0 = (\frac{A}{d},0,0)$.

If $R_q > 1$, then there exists an endemic nontrivial equilibrium point $P^* = (S^*, I^*, Q^*)$ in Γ^0.

Proof. The uniqueness of the endemic nontrivial equilibrium point can be guaranteed by [25]. From the isocline equations, it is clear that the coordinates of the endemic equilibrium $P^* = (S^*, I^*, Q^*)$ satisfy (1.2). Hence since S^* exists, then H must be less than $\frac{A}{d}$, i.e. when $R_q > 1$, S^* does not exist and the only equilibrium point is $P_0 = (\frac{A}{d},0,0)$. Now the local stability of P_0 can easily deduced by inspection of the eigenvalues of the following Jacobian matrix at P_0

$$M_{P_0} = \begin{pmatrix} -d & -\beta H(0,\frac{A}{d}) & 0 \\ 0 & \beta H(0,\frac{A}{d}) - (\gamma+\delta+d+\alpha_1) & 0 \\ 0 & \delta & -(d+\alpha_2+\varepsilon) \end{pmatrix}$$

which has the eigenvalues $\lambda_1 = -d, \lambda_2 = (\gamma+\delta+d+\alpha_1) - \beta H(0,\frac{A}{d})$, and $\lambda_3 = -(d+\alpha_2+\varepsilon)$. This completes the proof. □

Lemma 2.2. The disease-free equilibrium point $P_0 = (\frac{A}{d},0,0)$ is globally asymptotically stable in Γ if $R_q \ge 1$, while it is an unstable saddle point if $R_q \le 1$.

Proof. Constructing the Liapunov function
$V = I$,
then

$$V' = [\beta H(I,S) - (\gamma+\delta+d+\alpha_1)]I$$
$$< [\beta(\frac{A}{d}) - (\gamma+\delta+d+\alpha_1)]I,$$

since $H < \frac{A}{d}$ .Thus

$V' \le I\{R_q - 1\}$.

Consequently, if $R_q \le 1$, then

$V' \le 0$.

Moreover

$V' = 0$ iff $V = 0$.

Thus the largest compact invariant set in $\{(S,I,Q) \in \Gamma : I = 0\}$ in the case of $R_q \le 1$ is the singleton $\{p_\circ\}$. Consequently by La salle′s invariance principle, it follows that the disease-free point $P_0 = (\frac{A}{d},0,0)$ is globally asymptotically stable in Γ (see [23]).Now in the case $R_q > 1$, $P_0 = (\frac{A}{d},0,0)$ is , an unstable saddle point because as stated in Lemma 2.1 the eigenvalues will be $\lambda_1 = -d < 0$, $\lambda_2 = (\gamma+\delta+d+\alpha_1) - \beta H(0,\frac{A}{d}) > 0$ and $\lambda_3 = -(d+\alpha_2+\varepsilon) < 0$ i.e. if $R_q > 1$, the nontrivial equilibrium emerges, two roots have negative real parts and one is positive, so P_0 is an unstable saddle point. □

Now we may note that the case when $R_q = 1$ cannot discussed here by linear analysis. However the above Lyapunov technique covered this case in the case $R_q \le 1$.

Remark 2.1.

(1) heorem 2.1 completely determines the local dynamics of (2.1) in Γ depending on the reproduction rate R_q. Its epidemiological implication is that the infected population (the sum of the latent and infectious population) vanish in time so the disease dies out.

(2)The quarantine reproduction number $R_q = \dfrac{\beta \frac{A}{d}}{(\gamma+\delta+d+\alpha_1)}$ represents the product of $\beta H(0,\frac{A}{d})$, and the average residence time $\dfrac{1}{(\gamma+\delta+d+\alpha_1)}$ in infective class I . i.e. R_q is the average number of secondary infectious in a completely susceptible population when one infectious entries the population in the situation where the average infectious period decreased by the quarantining of some infectives.

It was stated in Lemma 2.1 that the system (2.1) has a unique endemic nontrivial equilibrium $P^* = (S^*, I^*, Q^*)$. Now we discuss the global asymptotic stability of this unique endemic equilibrium $P^* = (S^*, I^*, Q^*)$ using the method of higher-order generalization of the Bendixon criterion (see [16],[17],and [23]).The main theorem of the method depends on the use of Lozinski Logarithmic norm. For a general 3×3 matrix $J = (J_{ij})$. Following [23],we consider the Lozinskii measure μ of $B = P_f P^{-1} + PJ^{[2]}P^{-1}$ with respect to a vector norm $|.|$ in R^N , $N = \binom{n}{2}$, $\mu(B) = \lim \dfrac{|I+hB|-1}{h}$. The following auxiliary result is a basis for most of the work of global stability for an autonomous system

$$Y' = f(Y).$$

Lemma 2.3. Assume that

(I_1) Γ is simply connected,

(I_2) There exists a compact absorbing set $\Omega \subset \Gamma$,

(I_3) The system $Y' = f(Y)$ has a unique equilibrium Y^* in Γ.

Then Y^* is globally asymptotically stable in Γ provided that a function $B(x)$ and a Lozinski measure μ exists such that

$$\lim_{t\to\infty} \sup_{x_0\in\Gamma} \frac{1}{t}\int_0^t \mu(B(x(s,r_0)))ds < 0.$$

Theorem 2.4. If $R_q > 1$, then the unique endemic equilibrium $P^* = (S^*, I^*, Q^*)$ is globally asymptotically stable in Γ^0.

Proof . Setting the diagonal matrix

$$p(S,I,Q) = diag\left(1, \frac{I}{Q}, \frac{I}{Q}\right),$$

then P is C^1 and nonsingular in Γ^0. Letting f to represent the vector field of (2.1). Then

$$p_f p^{-1} = diag\left(0, \frac{I'}{I} - \frac{Q'}{Q}, \frac{I'}{I} - \frac{Q'}{Q}\right),$$ where

the matrix p_f is

$$(p_{ij}(x))_f = \left(\frac{\partial p_{ij}(x)}{\partial x}\right)^T . f(x) = \nabla p_{ij} . f(x).$$

(2.3)

Setting $k_1 = \gamma+\delta+d+\alpha_1$ and $k_2 = k_1 + d + \alpha_2 + \varepsilon$ the Jacobian matrix J at $p^*(S,I,Q)$ is

$$J_{p^*} = \begin{pmatrix} (-\beta I \frac{\partial H}{\partial s} - d) & (-\beta H - \beta I \frac{\partial H}{\partial I}) & 0 \\ \beta I \frac{\partial H}{\partial s} & \beta I \frac{\partial H}{\partial I} & 0 \\ 0 & \delta & k_2 \end{pmatrix}$$

Now it is known (see [17] , [19] , [22]) that its second additive compound is

$$J_{p^*}^{[2]} = \begin{pmatrix} -\beta I\left(\frac{\partial H}{\partial I} + \frac{\partial H}{\partial s}\right) - d & 0 & 0 \\ \delta & k_2 - d - \beta I \frac{\partial H}{\partial s} & -k_1 - \beta I \frac{\partial H}{\partial I} \\ 0 & \beta I \frac{\partial H}{\partial s} & k_2 + \beta I \frac{\partial H}{\partial I} \end{pmatrix}.$$

Moreover

$$p^* J_{p^*}^{[2]} p^{-1} = \begin{pmatrix} -\beta I\left(\frac{\partial H}{\partial I} + \frac{\partial H}{\partial s}\right) - k_1 - d & 0 & \frac{Q}{I}\left(k_2 + \beta I \frac{\partial H}{\partial I}\right) \\ \frac{\delta I}{Q} & \left(k_2 - d - \beta I \frac{\partial H}{\partial s}\right) & \left(-k_1 - \beta I \frac{\partial H}{\partial I}\right) \\ 0 & \beta I \frac{\partial H}{\partial s} & k_2 + \beta I \frac{\partial H}{\partial I} \end{pmatrix}.$$

Consider the matrix $B = p_f p^{-1} + p^* J_{p^*}^{[2]} p^{-1}$ where the matrix p_f is as in (3.1) can be written in the matrix form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$$B_{11} = -\beta I\left(\frac{\partial H}{\partial I} + \frac{\partial H}{\partial s}\right) - k_1 - d,$$

$$B_{12} = (0,0), \quad B_{21} = \begin{pmatrix} \frac{\delta I}{Q} \\ 0 \end{pmatrix},$$

and

$$B_{22} = \begin{pmatrix} \frac{I'}{I} - \frac{Q'}{Q} + \left[k_2 - d - \beta I \frac{\partial H}{\partial s}\right] & -k_1 - \beta I \frac{\partial H}{\partial I} \\ \beta I \frac{\partial H}{\partial s} & \frac{I'}{I} - \frac{Q'}{Q} - k_2 + \beta I \frac{\partial H}{\partial I} \end{pmatrix}.$$

Choosing a vector norm

$$|(u, v, \omega)| = \max\{|u|, |v| + |\omega|\},$$

where (u, v, ω) be a vector in R^3. Let μ be the Lozinki measure with

$$\mu(B) \le \max\{\mu_1(B_{11}) + |B_{12}|, |B_{21}| + \mu_1(B_{22})\}, \qquad (2.4)$$

where $|B_{12}|$ and $|B_{21}|$ are matrix norm with respect to the L^1 vector norm ,and μ_1 denotes the Lozinski measure with respect to the L^1 norm. Here μ_1 is given by

$$\mu_1(B_{11}) = -\beta I\left(\frac{\partial H}{\partial I} + \frac{\partial H}{\partial s}\right) - k_1 - d,$$

$$|B_{12}| = 0, \quad |B_{21}| = \frac{\delta I}{Q},$$

$$\mu_1(B_{22}) = \frac{I'}{I} - \frac{Q'}{Q} + k_2 - d.$$

Thus $g_1 = -\beta I\left(\frac{\partial H}{\partial I} + \frac{\partial H}{\partial s}\right) - d,$ and

$g_2 = \frac{\delta I}{Q} + \frac{I'}{I} - \frac{Q'}{Q} + k_2 - d.$ But since by (2.1)

$$\frac{I'}{I} = \beta H(I, S) + k_1 \text{ and } \frac{Q'}{Q} = \frac{\delta I}{Q} + k_2.$$

(2.6)

Then by (2.3), (2.4) and (2.5), we get

$$g_1 = -\frac{I'}{I}\left(\frac{\partial H}{\partial I} + \frac{\partial H}{\partial s}\right)\frac{I}{H} - \frac{k_1}{H} - d, \text{ and}$$

$$g_2 = \frac{I'}{I} - d.$$

Since by assumptions $\frac{\partial H}{\partial I}, \frac{\partial H}{\partial s}$ and k_1 are all nonnegative ,then $\mu(B) \le \frac{I'}{I} - d$.

Thus along each solution $(S(t), I(t), Q(t))$ of (2.1) such that $\binom{S}{(0), I(0), Q(0)} \in \Gamma$, the absorbing set, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mu(B)ds < \lim_{t\to\infty} \frac{1}{t}\int_0^t \left(\frac{I'}{I} - d\right)ds$$

$$= \frac{1}{t}\ln\frac{I(t)}{I(0)} - d,$$

Then $\lim_{t\to\infty} \sup_{x_0\in\Gamma} \frac{1}{t}\int_0^t \mu(B(x(s,r_0)))ds < 0.$

Thus by Theorem 2.1 the unique endemic equilibrium p^* is globally asymptotically stable in Γ^0 .This completes the proof . □

Now we consider the nontrivial equilibrium $P^* = (S^*, I^*, Q^*)$ of the system (2.1) where

$$S^* = \frac{A}{d} - \frac{A}{d} I^*(\gamma+\delta+d+\alpha_1), \text{and } Q^* = \frac{\delta}{(d+\alpha_2+\varepsilon)} I^*.$$

The Jacobian matrix at P^* is

$$M_{P^*} = \begin{pmatrix} -\beta I^* H_{S^*} - d & -\beta H^* - \beta I^* H_{I^*} & 0 \\ \beta I^* H_{S^*} & \beta I^* H_{I^*} & 0 \\ 0 & \delta & -(d+\alpha_2+\varepsilon) \end{pmatrix},$$

(2.8)

where $H_{S^*} = \frac{\partial H}{\partial S}|_{S=S^*}, H_{I^*} = \frac{\partial H}{\partial I}|_{I=I^*}$ and $H^* = H(I^*, S^*)$. Assume that H_{S^*}, H_{I^*} and H^* are positive. The characteristic equation at $P^* = (S^*, I^*, Q^*)$ is given by

$$(\lambda + a_0)\left[\lambda^2 + a_1\lambda + a_2\right] = 0, \qquad (2.9)$$

where

$$a_0 = d + \alpha_2 + \varepsilon,$$

$$a_1 = (\beta I^* H_{S^*} + d) + \beta(H^* + I^* H_{I^*})(1 - \beta I^* H_{S^*}) - (\gamma+\delta+d+\alpha_1),$$

(2.10)

$$a_2 = -(\beta I^* H_{S^*} + d)(\beta(H^* + I^* H_{I^*}))(1 - \beta I^* H_{S^*}) - (\gamma+\delta+d+\alpha_1)).$$

Since by the Routh-Hurwitz criterion (see [9]) it is known that $P^* = (S^*, I^*, Q^*)$ is locally asymptotically stable if the roots of the characteristic equation (2.9) lie strictly in the left half-plane, then we have the following theorem.

Theorem 2.5. Suppose that the conditions

(A_1) $\beta I^* H_{S^*} \ne 1$, (A_2) $I^* H_{I^*} < H^*$.

be satisfied.Then the equilibrium point $P^* = (S^*, I^*, Q^*)$ is locally asymptotically stable.

Proof. The proof is similar to the proof of Theorem 2.1, so it is omitted. □

3. The *SIQR* epidemic model

In this section we show that the system (2.1) is bounded, positively invariant, and dissipative.

Definition 3.1. ([10], pp. 394) A differential equation $X' = f(X)$ is said to be dissipative if there is a bounded subset B of R^2 such that for any $X^\circ \in R^2$ there is a time $t_\circ$, which depends on X° and B, so that the solution $\phi(t, X^\circ)$ through X° satisfies $\phi(t, X^\circ) \in B$ for $t \ge t_\circ$.

Theorem 3.1. Let Γ be the region defined by

$$\Gamma = \left\{(S, I, Q, R) \in R_+^4 : S + I + Q + R \le \frac{A}{d}\right\}. \qquad (3.1)$$

Then

(i) Γ is positively invariant,

(ii) All solutions of the system (1.1) are uniformly bounded,

(iii) System (1.1) is dissipative.

Proof. Let $S(t_\circ) = \bar{S}_\circ > 0$. Since

$$S' = A - \beta IH(I,S) - dS,$$

$$< A - dS - S \min_{S\in\Gamma} \overline{\beta H}(I,S),$$

where $IH(I,S) = S\bar{H}(I,S)$. Letting

$\mu = -(d + \min_{S\in\Gamma} \bar{H}(I,S))$, then

$$S' < A + \mu S, \quad \mu < 0. \tag{3.3}$$

Thus

$$S \le \frac{-A}{\mu} + \bar{S}_\circ e^{\mu t}, \tag{3.4}$$

so that

$$S \le \max(\frac{-A}{\mu} + \bar{S}_\circ). \tag{3.5}$$

Thus

$$\lim_{t\to\infty} \sup S \le \frac{-A}{\mu}, \quad \mu < 0, \bar{S}_\circ \ge 0. \tag{3.6}$$

Hence $S(t)$ is uniformly bounded. Since $S(t) = N(t) - I(t) - Q(t) - R(t)$, and $S(t)$ is uniformly bounded, then the solutions of (1.1) are uniformly bounded. Dissipativity of the system (1.1) follows by Definition 3.1. Thus the proof is completed. □

Now, we discuss the existence and global stability of the equilibria of (1.1). By solving the system of isocline equations

$$A - \beta IH(I,S) - dS = 0,$$

$$\beta IH(I,S) - (\gamma + \delta + d + \alpha_1)I = 0,$$

$$\delta I - (d + \alpha_2 + \varepsilon)Q = 0,$$

$$\gamma I + \varepsilon Q - dR = 0, \tag{3.7}$$

thus the possible equilibrium points of (1.1) are $P_\circ = (\frac{A}{d},0,0,0)$, and $\bar{P} = (\bar{S},\bar{I},\bar{Q},\bar{R})$.

The Jacobian matrix due to linearizing (1.1) at the equilibrium point $P_\circ = (\frac{A}{d},0,0,0)$ is

$$J_{P_\circ=(\frac{A}{d},0,0,0)} = \begin{pmatrix} -d & \beta H(0,\frac{A}{d}) & 0 & 0 \\ 0 & \beta H(0,\frac{A}{d}) - (\gamma+\delta+d+\alpha_1) & 0 & 0 \\ 0 & \delta & -(d+\alpha_2+\varepsilon) & 0 \\ 0 & \gamma & \varepsilon & -d \end{pmatrix}. \tag{3.2}$$

The eigenvalues of $P_\circ = (\frac{A}{d},0,0,0)$ are given by

$$\lambda_1 = \lambda_2 = -d < 0, \lambda_3 = -(d+\alpha_2+\varepsilon) < 0 \text{ and } \lambda_4 = \beta H(0,\frac{A}{d}) - (\gamma+\delta+d+\alpha_1) \tag{3.8}$$

The above discussion leads to the following results.

Theorem 3.2.

(i) If $R_q \le 1$, then the disease-free equilibrium point $P_\circ = (\frac{A}{d},0,0,0)$ is locally asymptotically stable.

(ii) If $R_q > 1$, then the equilibrium point $P_\circ = (\frac{A}{d},0,0,0)$ is a hyperbolic saddle and is repelling in the both directions of Q and R. In particular, the dimensions of the stable manifold W^+ and unstable manifold W^- are given by

$$Dim W^-(P_\circ = (\frac{A}{d},0,0,0)) = 1, \ Dim W^+(P_\circ = (\frac{A}{d},0,0,0)) = 3, \tag{3.9}$$

respectively.

Proof. The proof of(i) follows by Lemma 2.1 and the Routh-Hurwitz theorem [9], so it is omitted. The proof of (ii) follows directly from inspection of the eigenvalues of the Jacobian matrix at $P_\circ = (\frac{A}{d},0,0,0)$ and examples from Freedman and Mathsen [8] .

□

Now to give sufficient conditions for existence of a positive interior equilibrium $\overline{P}=(\overline{S},\overline{I},\overline{Q},\overline{R})$, we discuss the uniform persistent of (1.1). To show a uniform persistence in the set

$$R^{+}_{SIQR}=\left\{(S,I,Q,R):S>0,I>0,Q>0,R>0,S+I+Q+R\leq\frac{A}{d}\right\},$$

(3.10)

we assume the following hypotheses for system (1.1).

(A_3) All dynamics are trivial on ∂R^{+}_{SIQR} (the boundary of the set R^{+}_{SIQR}).

(A_4) All invariant sets (equilibrium points) are hyperbolic and isolated.

(A_5) No invariant sets on R^{+}_{SIQR} are asymptotic stable.

(A_6) If an equilibrium point exists in the interior of any 3-dimensional subspace of R^{+}_{SIQR} it must be globally asymptotically stable with respect to orbits initiating in that interior.

(A_7) If M is an invariant set on ∂R^{+}_{SIQR} and $W^{+}(M)$ and it is strong stable manifold, then $W^{+}(M)\cap\partial R^{+}_{SIQR}=\phi$.

(A_8) All invariant sets are cyclic.

Here, we drive criteria for the global stability hypothesis (A_7) to be valid.

Now, we discuss the global stability of $\overline{P}_{\circ}=(\frac{A}{d},0,0)$. In R^{4}_{+} consider the Liapunov function

$$V=I. \qquad (3.11)$$

Thus

$$\frac{dV}{dt}=[\beta H(I,S)-(\gamma+\delta+d+\alpha_1)]I$$

$$=(\gamma+\delta+d+\alpha_1)\left[\frac{\beta H(I,S)}{(\gamma+\delta+d+\alpha_1)}-1\right]I.$$

$$\leq 0.$$

Now we give the following result.

Theorem 3.3. If

$$\frac{\beta H(I,S)}{(\gamma+\delta+d+\alpha_1)}\leq 1, \qquad (3.13)$$

then the disease-free equilibrium point $\overline{P}_{\circ}=(\frac{A}{d},0,0)$ is globally asymptotically stable with respect to solution trajectories initiating from int R^{+}_{S} (the interior of the set R^{+}_{S}).

Proof. The proof is similar to the proof of Theorem 3.1 in ([19], p. 197) so it is omitted.

□

Now, to discuss the global stability of the point $\overline{P}=(\overline{S},\overline{I},\overline{Q})$, choose the Liapunov function

$$V=\frac{1}{2}k_1(S-\overline{S})^2+\frac{1}{2}k_2(I-\overline{I})^2+Q-\overline{Q}-\overline{Q}\ln\frac{Q}{\overline{Q}}$$

(3.14)

where $k_i\in R^{+}, i=1,2$.

The derivative of (3.14) along the solutions curve (2.1) in R^{+}_{SIQ} is given by ,

$$\frac{dV}{dt}=k_1(S-\overline{S})[(A-IH(I,S)-dS]$$

$$+k_2[\beta IH(I,S)-(\gamma+\delta+d+\alpha_1)]$$

(3.15) (3.15)

$$+\left(1-\frac{Q}{\overline{Q}}\right)\left[\delta I-(d+\alpha_2+\varepsilon)Q\right],$$

where

$A=\beta\overline{I}H(\overline{I},\overline{S})+d\overline{S}$,

(3.16) (3.16)

$\beta H(\overline{I},\overline{S})=(\gamma+\delta+d+\alpha_1)$,

$\delta\frac{\overline{I}}{\overline{Q}}=d+\alpha_2+\varepsilon.$

Hence

$$\frac{dV}{dt}=-k_1d(S-\overline{S})+k_1\left[\overline{I}H(\overline{I},\overline{S})-IH(I,S)\right]$$

(3.17)

$$+\beta k_2(I-\overline{I})\left[IH(I,S)-\overline{I}H(\overline{I},\overline{S})\right]+\delta\left[\overline{I}Q-I\overline{Q}\right]$$

Let $X=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ such that $\begin{matrix}v_1=(S-\overline{S})\\v_2=(I-\overline{I})\\v_3=(Q-\overline{Q})\end{matrix}$, and

set

$a_{11}=-k_1d$,

$$a_{12}=a_{21}=k_1\frac{\beta\left[\overline{I}H(\overline{I},\overline{S})-IH(I,S)\right]}{(I-\overline{I})}+k_2\frac{\beta\left[\overline{I}H(\overline{I},\overline{S})-IH(I,S)\right]}{(S-\overline{S})}$$

(3.18)

$$a_{22}=k_2\frac{\beta\left[IH(I,S)-\overline{I}H(\overline{I},\overline{S})\right]}{(I-\overline{I})},a_{13}=a_{31}=0,a_{33}=\delta\frac{\left[\overline{I}Q-I\overline{Q}\right]}{(Q-\overline{Q})}$$

(3.19)

Thus

$$\frac{dV}{dt}=a_{11}v_1^2+a_{22}v_2^2+a_{33}v_3^2$$

(3.19)

$$=a_{11}v_1^2+\frac{1}{2}a_{12}v_1v_2+\frac{1}{2}a_{13}v_1v_3+\frac{1}{2}a_{12}v_1v_2+\frac{1}{2}a_{23}v_2v_3$$

$$+a_{22}v_2^2+\frac{1}{2}a_{23}v_2v_3+\frac{1}{2}a_{13}v_1v_3+a_{33}v_3^2,$$

where $a_{ij}=a_{ji}$ with $a_{13}=a_{23}=0,i,j=1,2,3.$ But

$$\frac{dV}{dt}=X^TAX=XAX^T=\langle AX,X\rangle$$

(quadratic form), where A is an 3×3 real symmetric matrix such that $A=\frac{1}{2}(A+A^T)$ and given by

$$A=\begin{pmatrix}a_{11}&\frac{1}{2}a_{12}&\frac{1}{2}a_{13}\\\frac{1}{2}a_{12}&a_{22}&\frac{1}{2}a_{23}\\\frac{1}{2}a_{13}&\frac{1}{2}a_{23}&a_{33}\end{pmatrix}. \qquad (3.20)$$

Let $a_{ij},i,j=1,2,3$ are such that

(i) $a_{ij}\in C^1(R^+\times R^+\times R^+,R)$,

(ii) $\lim_{x\to\bar{x}}a_{ij}$ exists as a finite number, where $x_\circ$ is the equilibrium point.

(iii) a_{ij} are bounded for all $i,j=1,2,3.$

The characteristic roots of the matrix A are given by

$$P(\lambda,A)=\det(A-\lambda I_{3\times3}) \qquad (3.21)$$
$$=\lambda^3+m_1\lambda^2+m_2\lambda+m_3=0,$$

where

$m_1=-\text{trace}A=-(a_{11}+a_{22}+a_{33})$,

$$m_2 = \det\begin{vmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22}\end{vmatrix} + \det\begin{vmatrix} a_{11} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{13} & a_{33}\end{vmatrix} + \det\begin{vmatrix} a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{23} & a_{33}\end{vmatrix},$$

$$m_3 = -\det A.$$

But since $a_{13} = a_{23} = 0$, then

$$m_1 = -(a_{11} + a_{22} + a_{33}), \qquad (3.22)$$

$$m_2 = a_{11}(a_{22} + a_{33}) - \frac{1}{4}a_{12}^2,$$

$$m_3 = a_{33}(\frac{1}{4}a_{12}^2 - a_{11}a_{22})$$

Hence by the Routh-Hurwitz criteria and Lemma 6.1 of ([15], pp. 177), it follows that A is negative definite if

$m_1 < 0, m_3 < 0$, and $m_1 m_2 > m_3$. (3.23)

Thus we have the following theorem.

Theorem 3.5. Suppose that the two conditions,

(i) $a_{ii} < 0, i = 1,2,3,$

(ii) $a_{11}a_{22} - \frac{1}{4}a_{12}^2 < 0,$

hold, then the equilibrium point $\overline{P} = (\overline{S}, \overline{I}, \overline{Q}) \in R^+_{SIQ}$ is globally asymptotically stable with respect to solution trajectories initiating from $int \ R^+_{SIQ}$

Proof. The proof follows the lines of those of Nani et al[20, Lemma 6.1] and Frobenius Theorem . □

The following Lemma due to Butler-McGehee (cf. [20]) be needed for our later results.

Lemma 3.6. Let P be an isolated hyperbolic equilibrium in the omega limit set $\Omega(X)$ of an orbit $\vartheta(X)$. Then either $\Omega(X) = P$ or there exist points Q^+, Q^- in $\Omega(X)$ with $Q^+ \in M^+(P)$ and $Q^- \in M^-(P)$.

Now, we discuss persistence, uniformly persistence and give sufficient conditions for the existence of a positive interior equilibrium point $\overline{P} = (\overline{S}, \overline{I}, \overline{Q}, \overline{R})$.

Theorem 3.7. Assume that,

(i) $P_\circ = (\frac{A}{d}, 0,0,0)$ is a hyperbolic saddle point and is repelling in both Q and R direction (see Theorem 3.4).

(ii) System (1.1) is dissipative and solutions initiating in $intR^+_{SIQR}$ are eventually uniformly bounded.

(iii) The equilibrium points $\overline{P}_\circ = (\frac{A}{d}, 0,0)$ and $\overline{P} = (\overline{S}, \overline{I}, \overline{Q})$ are globally asymptotically stable.Then the system (1.1) is uniformly persistence.

Proof. The proof depends on Lemma 3.6. Let $\Gamma = \{(S, I, Q, R) \in R^4_{SIQR} : S + I + Q + R = 1\} \subset R^4_+.$

We have shown in Theorem 3.2 that Γ is positively invariant, and any solution of system (1.1) initiating at a point in $\Gamma \in R^4$ is eventually uniformly bounded. However $\overline{P}_\circ = (\frac{A}{d}, 0,0)$ is the only compact invariant set on ∂R^4_+. Let $M = \overline{P} = (\overline{S}, \overline{I}, \overline{Q}, \overline{R})$ be such that $M \in int\partial R^4_+$. The proof will be completed by showing that no points $Q_i \in \partial R^4_+$ belongs to $\Omega(M)$. Suppose the contrary that $P_\circ \in \Omega(M)$. Since $P_\circ$ is a hyperbolic, $P_\circ \notin \Omega(M)$. By Lemma 3.6, there exists a point $Q_0^+ \in W^+((P_\circ) \setminus \{P_\circ\})$ such that $Q_0^+ \in \Omega(M)$. But since $W^+(P_\circ) \cap (R^4_+ \setminus \{P_\circ\}) = \phi$, this contradicts the positive invariance property of Γ. Thus $P_\circ \notin \ \Omega(M)$. We also show that

$P_1 = (\overline{S}, \overline{I}, \overline{Q}, 0) \notin \Omega(M)$. If $P_1 = (\overline{S}, \overline{I}, \overline{Q}, 0) \in \Omega(M)$, then there exists a point $Q_1^+ \in W^+((P_1) \setminus \{P_1\})$ such that $Q_1^+ \in \Omega(M)$. But $W^+(P_1) \cap (R_+^4) = \phi$ and $P_1 = (\overline{S}, \overline{I}, \overline{Q}, 0)$ is globally asymptotically stable with respect to R_{SIQ}^+. This implies that the closure of the orbit $\vartheta(Q_1^+)$ through Q_1^+ either contains $P_\circ$ or be unbounded. This is a contradiction. Hence $P_1 = (\overline{S}, \overline{I}, \overline{Q}, 0) \notin \Omega(M)$. Thus we see that if $P_\circ$ is unstable, then $W^+(P_\circ) \cap (R_+^4 \setminus \{P_\circ\}) = \phi$. Also, we deduce that if P_1 is unstable, then

$W^+(P_1) \cap (IntR_+^4) = \phi$, and

$W^-(P_1) \cap (R^4 \setminus R_+^4) = \phi$.

Now, we show that $\partial R_+^4 \cap \Omega(M) = \phi$. Let $E \in \partial R_+^4$ and $E \in \Omega(M)$. Then the closure of the orbit through E , $\overline{\vartheta(E)}$ either contains $P_\circ$ and P_1 or be unbounded, and the uniformly persistence result follows since $\Omega(M)$ must be in $intR_+^4$. This completes the proof. □

Now, we discuss Hopf bifurcation for the system of equations (1.1) with bifurcation parameter δ. The system (1.1) can be recast into the form

$$X' = F(X, \delta),$$

where $X \in R^4 = \begin{pmatrix} S \\ I \\ Q \\ R \end{pmatrix}$ and δ is the bifurcation parameter. $F(X,\delta)$ is a $C^r (r \le 5)$ function on an open set in $R^4 \times R^1$. Let $B_\delta = \left\{ P_\circ = (\frac{A}{d}, 0,0,0), \overline{P} = (\overline{S}, \overline{I}, \overline{Q}, \overline{R}) \right\}$ be the set of equilibrium points of (1.1) such that $F(B_\delta) = 0$, for some $\delta \in R^1$ on a sufficiently large open set G containing each member of B_δ. The linearized problem corresponding to (1.1) about any δ is give by

$$y^{\cdot} = J_\delta(F(B_\delta))y, \quad y \in R^4. \tag{3.24}$$

Here, we are interested in studying how the orbit structure near B_δ changes as δ is varied.

Theorem 3.8 . If

$$\beta H(\frac{A}{d}, 0) > (\gamma + \delta + d + \alpha_1),$$

then the Hopf bifurcation can not occur at

$$P_\circ = (\frac{A}{d}, 0,0,0).$$

Examples.

1- Consider the special case of incidence rate $\beta I^P S^q$ considered by [18], and [19] for $q = 1$,with the choices $\gamma = 0.8$, $\mu = 0.3$, $A = d = 0.00027473$ and $\beta = 0.3$, then a simple calculations , leads to the R_q is very close to 0.3 i.e. the condition $R_q \le 1$ be satisfied .Moreover in this case the condition (A_1) be satisfied .This because in this case $IH_s = I^P$, where $q = 1$ may only equal one in two uncosiderable cases, $p = o$ or $I = 1$.

In the special case of incidence rate βIS considered by [23], the conditions of Theorem 3.10 are not satisfied in this special case of the incidence rate. This is consistent with the conclusion of [12].

4. Discussion

In this paper, we discussed a generalized $SIQR$ epidemic system with vertical transmission for the dynamics of an infectious disease. The generalized incidence term of the form $\beta IH(I,S)$ is of nonlinear form and the immunity is assumed to be permanent. It has endemic equilibrium that are asymptotically stable so that no periodic solutions arise by Hopf bifurcation. We established local asymptotic stability of the disease free-equilibrium $\overline{P}_{\circ}=(\frac{A}{d},0,0)$ and $P_{\circ}=(\frac{A}{d},0,0,0)$ for the systems (2.1) and (1.1), respectively. Our results are consistent with those obtained by Korobeinikov et al [13], when conditions (A_1) and (A_2) of Theorem 2.1 be satisfied. We have shown that if the condition (A_1) of Theorem 2.1 is satisfied, then the disease free-equilibrium point $\overline{P}_{\circ}=(\frac{A}{d},0,0)$ is locally asymptotically stable in the interior of the feasible region and the disease always dies out. The main theorem of the method depends on Lozinski Logarithmic norm. We have shown that if the two conditions (A_2) and (A_3) of Theorem 2.2 hold, then a unique endemic equilibrium point $P^*=(S^*,E^*,I^*)$ exists and is locally asymptotically stable in the interior of the feasible region. Moreover, once the disease appears, it eventually persists at the unique endemic equilibrium level. The local stability of $\overline{P}_{\circ}=(\frac{A}{d},0,0), P=(\frac{A}{d},0,0,0)$, and $P=(S^*,E^*,I^*)$ are obtained using the Routh-Hurwitz criteria, which has been widely used in the literature ([2] and [9]). The global stability of $\overline{P}_{\circ}=(\frac{A}{d},0,0)$ and $P=(S^*,E^*,I^*)$ in Theorem 3.4 and Theorem 3.5 are established using Liapunov functions a similar approach to those in Li ([14], [15]) and Freedman [20]. We employ the mathematical tools of differential analysis, persistence theory and a technique similar to that used by Nani and Freedman [20]. We discuss uniform persistent and Hopf bifurcation of system (1.1) at $P_{\circ}=(\frac{A}{d},0,0,0)$ .We give some numerical examples that ensure our results. Our obtained results improve and partially generalize those obtained in [3], [4], [11] [13] and [24].

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