Bitopological spaces via Double topological spaces

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Abstract: In this paper we shall study some bitopological properties via double topological spaces. We characterize the notions of pairwise continuous (resp. pairwise open, pairwise closed)(P .continuous, P - open, P - closed, for short) by a double continuous (resp. double open, double closed) mappings between double topological spaces. Also, we characterize the notions of P^* - continuous (resp. P^* -open, P^* - closed) by a supra double continuous (resp. open, closed) mappings between supra double topological spaces. Finally, we investigate the relationships between these types of mappings and give some counter examples.
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1. Introduction

The notion of a bitopological space (X , τ_1, τ_2), that is a set X equipped with two topologies τ_1 and τ_2 was formulated by J. C. Kelly in [12].

There are several hundred works indicated to the investigation of bitopological spaces. The book [1] is a versatile introduction to the theory of bitopological spaces and its applications.

Flou set stems from some linguistic considerations of Yves Gentilhomme about the vocabulary of a natural language [6]. E. E. Kerre [13] introduced the mathematical definition of flou sets and binary operations on it.

In this paper we follow the suggestion of J. G. Garcia and S. E. Rodabaugh [5] that "double fuzzy set" is a more appropriate name 'than "intuitionistic fuzzy set", and therefore adopt the term "double-set" for the flou set, and "double-topology" for the flou topology.

There are several hundred works indicated to the investigation of double topology (eg [9, 10, 11, 15])

In this paper, making use the relation between bitopological spaces (BTS`s for short) and double topological spaces (DTS, for short), we characterize the notions of P - continuous (resp. P -open, P - closed) mappings by a double continuous (resp. open, closed) mappings.

Also, we introduce the notion of supra double topological space (SDTS, for short) and characterize the notion of P^* - continuous (resp, P^* -open, P^* - closed) mappings by supra double continuous (resp, open, closed) mappings.

Finally, we investigate the relationship between these types of mappings and give same counter examples.

Note that for the concepts and results that are used but not stated here we refer to [[2], [8], [14]].

1. Preliminaries:

In this section we shall present the fundamental definitions and concepts which will be needed in the sequel.

Definition 2.1.[9] i) A double set $\underline{A}$ (D- set for short) is an ordered pair

$(A_1, A_2) \in P(X) \times P(X)$ such that $A_1 \subseteq A_2$.

ii) The family of all double sets on X, will be denoted by D(X), i.e.

$$D(X) = \{(A_1, A_2) \in P(X) \times P(X): A_1 \subseteq A_2\}.$$

iii) The double set $\underline{X}$ =(X, X) is called the universal double set, and $\underline{\varphi}$ = (φ , φ) is called the empty double set.

Definition 2.2. [9] Let $\underline{A}$ =(A_1 , A_2), $\underline{B}$ =(B_1, B_2) $\in$ D (X). Then:

1) $\underline{A} \subseteq \underline{B} \Leftrightarrow A_i \subseteq B_i$, i =1,2

2) $\underline{A} = \underline{B} \Leftrightarrow A_i = B_i$, i =1, 2

3) $\underline{A} \cup \underline{B} = (A_1 \cup B_1, A_2 \cup B_2)$

4) $\underline{A} \cap \underline{B}$=($A_1 \cap B_1$, $A_2 \cap B_2$)

5) $\underline{A}^C$ =(A_2^C , A_1^C) where $\underline{A}^C$ is the complement of $\underline{A}$.

6) Let $\eta_1, \eta_2 \subseteq$ P(X). The double product of η_1 and η_2 is denoted by $\eta_1 \hat{\times} \eta_2$ and is defined by $\eta_1 \hat{\times} \eta_2$ = {(A_1, A_2)∈ $\eta_1 \times \eta_2 : A_1 \subseteq A_2$ }.

Definition 2.3.[9] Let f : X$\longrightarrow$Y be a mapping, and let $\underline{A}$∈ D(X) and $\underline{B}$∈ D(Y).

The image of $\underline{A}$, defined by f ($\underline{A}$)=(f (A_1), f (A_2)) and the preimage f^{-1} ($\underline{B}$) is defined by f^{-1} ($\underline{B}$) = (f^{-1} (B_1), ( )).

Definition 2.4 [9] Let X be a non- empty set. Then a family D (X) is called a double topology on X iff it satisfies the following axioms:

1) = (,), = (X, X)∈ η.

2) If $\underline{A}$, $\underline{B} \in \eta$, then $\underline{A} \cap \underline{B} \in \eta$.

3) If {$\underline{A}_s$: s∈ S}$\subseteq \eta$, then $\bigcup_{s \in S} \underline{A}_s \in \eta$.

If η satisfies the axioms (1, 3), then it is called a supra double topology.

The pair (X, η) is called a double topological space. Each member of η is called an open double set in X. The complement of an open double set is called a closed double set. For any $\underline{A}$∈ D (X), the double closure of $\underline{A}$ is denoted by $\overline{\underline{A}}$ and is defined by $\overline{\underline{A}} = \bigcap$ { $\underline{B}$ | $\underline{B} \in \eta^C$ and $\underline{A} \subseteq \underline{B}$ }.

Definition 2.5 [9] A mapping f : (X ,η)$\longrightarrow$(Y,θ) is called:

i) doubl continuous (D continuous for short) iff f^{-1}($\underline{B}$)∈η whenever $\underline{B}$∈θ.

ii) doubl open (D open for short) iff f ($\underline{A}$)∈ θ whenever $\underline{A}$∈ η.

iii) doubl closed (D closed for short) iff f ($\underline{A}$)∈ θ^C whenever $\underline{A}$∈ η^C.

Remark: [11] Every DTS (X ,η) define a BTS which is (X , π_1 , π_2) where π_1={$V_1 \subseteq X$:∃ $V_2 \subseteq X$ s.t (V_1, V_2) ∈ η } and π_2={$V_2 \subseteq X$:∃ $V_1 \subseteq X$ S.T (V_1, V_2) ∈ η }.

Conversely, every BTS (X ,τ_1, τ_2) define a DT

$\tau_1 \hat{\times} \tau_2$= {($A_1$, A_2)∈ D (X): $A_1 \in \tau_1$, $A_2 \in \tau_2$} on X associated with τ_1, τ_2.

Theorem 2.6.[9] If f : (X, η)$\longrightarrow$(Y, η^*) is a D-continuous function, then f : (X, π_i)$\longrightarrow$(Y, $\pi^*{}_i$) i = 1, 2 are continuous functions.

Theorem 2.7.[9] Let (X, $\tau_1 \hat{\times} \tau_2$) be a DTS and (Y, η) be any DTS. Then

f : (X, $\tau_1 \hat{\times} \tau_2$) $\to$ (Y, η) is a D- continuous function iff

f : (X, τ_i) $\to$ (Y, π_i) i = 1, 2 are continuous functions.

Definition 2.8.[3] A mapping f : (X, τ_1, τ_2)$\longrightarrow$ (Y, θ_1, θ_2) is called P-continuous (resp, P -open, P-closed) if f is τ_i - θ_i continuous(resp, open, closed), i=1, 2.

Definition 2.9. [4] A mapping I : P(X) $\longrightarrow$ P(X) is called supra- interior operator iff it satisfies the following axioms :

1) I (X) = X.
2) $I\ (A) \subseteq A$.
3) $I\ (A \cap B) \subseteq I\ (A) \cap I\ (B)$.
4) $I\ (I\ (A)) = I\ (A)$.

Proposition 2.10. [4] Let (X, τ_1, τ_2) be a BTS. Then

i) $\tau^* = \tau_1 \sqcup \tau_2 = \{ U_1 \cup U_2 : U_i \in \tau_i \}$ is a supra topology on X and (X , τ^*) is the STS-associated to (X, τ_1, τ_2)

ii) The operator I : P(X) $\longrightarrow$ P(X), defined by I(A) = $A^{01} \cup A^{02}$, A^{0i} is the τ_i - interior of A, (i=1,2), is a supra operator in which $\tau^* = \{ A \subseteq$ X: A = I(A)}.

Proposition 2.11. [4] Let (X, τ_1, τ_2) be a BTS and (X , τ^*) its associated STS. Then C : P (X) $\longrightarrow$ P (X) defined by $C\ (A) = \overline{A}^1 \cap \overline{A}^2$.
is a supra- closure operator which induces the supra topology τ^*.

Remark 2.12 in [1], the author used the notion of pairwise open in a BTS, which means that: A is p-open $\Leftrightarrow$ $A = U_1 \cup U_2$, $U_i \in \tau_i$ (i=1,2). In [4, 7], we used the same notion under the name of P^* - open or supra open in (X , τ^*), where τ^* is a supra topology generated by τ_1 and τ_2 . We say that $A \subseteq X$ is P^* - open (supra open) $\Leftrightarrow$ $A \in \tau^*$.

Definition 2.13.[4] A mapping f : (X, τ_1, τ_2) $\longrightarrow$ (Y, θ_1, θ_2) is called

i) P^* -continuous iff f^{-1} (U) $\in \tau^*$ whenever U$\in \theta^*$.

ii) P^* -open iff $f\ (V) \in \theta^*$ whenever $V \in \tau^*$.

iii) P^* -closed iff $f\ (V) \in \theta^{*C}$ whenever $V \in \tau^{*C}$.

3. Operation on DTS and SDTS:

Definition 3.1. A mapping C : D(X) $\longrightarrow$ D(X) is a called double closure operator iff it satisfies the following axioms :

1) $C\ (\underline{\varphi}) = \underline{\varphi}$.
2) $\underline{A} \subseteq C\ (\underline{A})$.
3) $C\ (\underline{A} \cup \underline{B}) = C\ (\underline{A}) \cup C\ (\underline{B})$.
4) $C\ (C\ (\underline{A})) = C\ (\underline{A})$.

If C satisfies the axioms (1, 2, 4) and the following axiom, it is called a supra double closure operator:

3*) $C\ (\underline{A} \cup \underline{B}) \supseteq C\ (\underline{A}) \cup C\ (\underline{B})$.

Theorem 3.2. Let (X, τ_1 , τ_2) be a BTS. The operator I_{12} : D(X) $\longrightarrow$ D(X), defined by I_{12} ($\underline{A}$)=([$A_1 \cap A_2^{02}$]^{01}, A_2^{02}) $\forall$ $\underline{A} \in$ D (X), is a double interior operator which generates the double topology $\tau_1 \hat{\times} \tau_2$ on X.

Proof: As a sample, we prove the duality of the property ((3), definition 3.1) above, i.e. we prove that $I_{12}\ (\underline{A} \cap \underline{B}) = I_{12}\ (\underline{A}) \cap I_{12}\ (\underline{B})$. The proof of the other parts are similar.

`` $I_{12}$ `` is a well defined map since

$A_1 \subseteq A_2 \Rightarrow [A_1 \cap A_2^{02}]^{01} \subseteq A_2^{02}$

3) $I_{12}\ [\underline{A} \cap \underline{B}] = I_{12}\ (A_1 \cap B_1,\ A_2 \cap B_2)$= ([[$A_1 \cap B_1$] $\cap$ [$A_2 \cap B_2$]^{02}]^{01},

[$A_2 \cap B_2$]^{02})=($A_1^{01} \cap B_1^{01} \cap (A_2^{02})^{01} \cap (B_2^{02})^{01}$, $A_2^{02} \cap B_2^{02}$)

=([$A_1^{01} \cap (A_2^{02})^{01}$] $\cap$ [$B_1^{01} \cap (B_2^{02})^{01}$], $A_2^{02} \cap B_2^{02}$)

= ([$A_1 \cap A_2^{02}$]^{01}, A_2^{02}) $\cap$ ([$B_1 \cap B_2^{02}$]^{01}, B_2^{02})

= $I_{12}\ (\underline{A}) \cap I_{12}\ (\underline{B})$

Then I_{12} is a double interior operator and hence it generates a double topology η on X where

η ={ $\underline{A}$ | I_{12} ($\underline{A}$)= $\underline{A}$ }= { $\underline{A}$ | ([$A_1 \cap A_2^{02}$]^{01}, A_2^{02})= $\underline{A}$ }

={ $\underline{A}$ |($A_1^{01} \cap (A_2^{02})^{01}$, A_2^{02}) =(A_1 , A_2)}= { $\underline{A}$ | $A_1^{01} = A_1 \wedge A_2^{02} = A_2$ }

={ $\underline{A}$ | $A_1 \in \tau_1 \wedge A_2 \in \tau_2$ }= $\tau_1 \hat{\times} \tau_2$.

Corollary 3.3. Let (X, τ_1, τ_2) be any BTS. Then the operator

C_{12}: D (X) $\longrightarrow$ D (X) defined by: $C_{12}\ (\underline{A}) = (\overline{A_1}^2 , \overline{\overline{A_1}^2 \cup A_2}^1)$ $\forall$ $\underline{A} \in$ D (X) is a double closure operator generates the double topology $\tau_1 \hat{\times} \tau_2$ on X.

Theorem 3.4. Let (X, τ_1, τ_2) be a BTS and let (X, τ^*) its associated supra topological space. Then the operator I^*: D (X) $\rightarrow$ D(X) defined by

$I^*(\underline{A}) = (I(A_1), I(A_2))$, where $I(A_i) = A_i^{01} \cup A_i^{02}$ (i=1, 2), is a supra double interior operator such that $\tau_{I^*} = \tau^* \hat{\times} \tau^*$.

Proof: The proof that I^* is a supra-interior operator, follows from the definition of I^* and the fact that I is a supra –interior operator (prop. 2.9). For the proof of

$\tau_{I^*} = \tau^* \hat{\times} \tau^*$, let $\underline{A} = (A_1, A_2) \in \tau^* \hat{\times} \tau^*$. Then $A_i \in \tau^*$, (i= 1, 2) and $I(A_i) = A_i$. SO, $I^*(\underline{A}) = (I(A_1), I(A_2)) = \underline{A} \Rightarrow A \in \tau_{I^*} \Rightarrow \tau^* \hat{\times} \tau^* \subseteq \tau_{I^*}$.

Conversely, Let $\underline{A} \in \tau_{I^*}$. Then $I^*(\underline{A}) = \underline{A} \Rightarrow (I(A_1), I(A_2)) = (A_1, A_2)$. Hence $A_i \in \tau^*$ (i= 1,2) and therefore $\underline{A} = (A_1, A_2) \in \tau^* \hat{\times} \tau^*$. So $\tau_{I^*} \subseteq \tau^* \hat{\times} \tau^*$ and consequently $\tau_{I^*} = \tau^* \hat{\times} \tau^*$.

Corollary 3.5. Let (X, τ_1, τ_2) be a BTS. Then the operator

C^*: D (X) $\rightarrow$ D (X) such that $C^*(\underline{A}) = (C(A_1), C(A_2))$,where $C(A_i) = \overline{A}^1_i \cap \overline{A}^2_i$ (i=1, 2) is a supra double closure operator such that $\tau_{C^*} = \tau^* \hat{\times} \tau^*$.

Theorem 3.6. Every double closure operator C : D (X) $\rightarrow$ D (X) generates a BTS (X, τ_1, τ_2), where $\tau_i = \{A_i \subseteq X: C((A_1, A_2)^C) = (A_1, A_2)^C, (A_1, A_2) \in$ D (X)}, i=1,2.

Proof: Straightforward.

4. The relations between P. continuous (resp P. open, P. closed) mappings and Duble continuous (resp duble open, double closed) mappings:

In this section, we characterize the notion of P-continuous (resp P -open, P -closed) by a D-continuous (resp D-open, D-closed) mappings.

Theorem 4.1. A mapping f : (X, τ_1, τ_2) $\rightarrow$ (Y, θ_1, θ_2) is pairwise continuous iff

f : (X, $\tau_1 \hat{\times} \tau_2$) $\rightarrow$ (Y, $\theta_1 \hat{\times} \theta_2$) is double continuous.

Proof: Let f : (X, τ_1, τ_2) $\rightarrow$ (Y, θ_1, θ_2) be P-continuous and

$\underline{B} = (B_1, B_2) \in \theta_1 \hat{\times} \theta_2$. Then $f^{-1}(B_1) \in \tau_1$ and $f^{-1}(B_2) \in t_2$. So,

$f^{-1}(\underline{B}) = (f^{-1}(B_1), f^{-1}(B_2)) \in \tau_1 \hat{\times} \tau_2$.

Hence f : (X, $\tau_1 \hat{\times} \tau_2$) $\rightarrow$ (Y, $\theta_1 \hat{\times} \theta_2$) is double continuous.

Conversely: Let f : (X, $\tau_1 \hat{\times} \tau_2$) $\rightarrow$ (Y, $\theta_1 \hat{\times} \theta_2$) be D-continuous and let $G_1 \in \theta_1$.

Then $(G_1, Y) \in \theta_1 \hat{\times} \theta_2$. So, $f^{-1}(G_1, Y) = (f^{-1}(G_1), X) \in \tau_1 \hat{\times} \tau_2$. Hence $f^{-1} G_1 \in \tau_1$ and f : (X, τ_1) $\rightarrow$ (Y, θ_1) is continuous function. Also, let $G_2 \in \theta_2$. Then $(\varphi, G_2) \in \theta_1 \hat{\times} \theta_2$. So $f^{-1}(\varphi, G_2) = (\varphi, f^{-1} G_2) \in \tau_1 \hat{\times} \tau_2$ .Hence $f^{-1} G_2 \in \tau_2$ and f : (X, τ_2) $\rightarrow$ (Y, θ_2) is continuous function. Therefore

f : (X, τ_1, τ_2) $\rightarrow$ (Y, θ_1, θ_2) is P - continuous.

Theorem 4.2. Let f : (X, τ_1, τ_2) $\rightarrow$ (Y, θ_1, θ_2) be a mapping. Then the following conditions are equivalent:

1) f : (X, $\tau_1 \hat{\times} \tau_2$) $\rightarrow$ (Y, $\theta_1 \hat{\times} \theta_2$) is double continuous.

2) $f^{-1}(\underline{B}) \in (\tau_1 \hat{\times} \tau_2)^C \ \forall \underline{B} \in (\theta_1 \hat{\times} \theta_2)^C$

3) $f(C_{12}(\underline{A})) \subseteq C_{12}(f(\underline{A})) \ \forall \underline{A} \in$ D (X)

4) $C_{12}(f^{-1}(\underline{B})) \subseteq f^{-1}(C_{12}(\underline{B}))$ $\forall \underline{B} \in$ D (Y)

5) $f^{-1}(I_{12}\,\underline{B}) \subseteq I_{12}\, f^{-1}(\underline{B})$ $\forall \underline{B} \in$ D (Y)

Proof: (1)$\longrightarrow$(2): Let $\underline{B}=(B_1, B_2) \in (\theta_1 \hat{\times} \theta_2)^C$.

Then $(B_1, B_2)^C \in \theta_1 \hat{\times} \theta_2 \Rightarrow$ $(B_2^{\,C}, B_1^{\,C}) \in \theta_1 \hat{\times} \theta_2 \Rightarrow f^{-1}(B_2^{\,C}, B_1^{\,C}) \in \tau_1 \hat{\times} \tau_2 \Rightarrow$ $(f^{-1}(B_2^{\,C}), f^{-1}(B_1^{\,C})) \in \tau_1 \hat{\times} \tau_2 \Rightarrow ([f^{-1}(B_2)]^C, [f^{-1}(B_1)]^C) \in \tau_1 \hat{\times} \tau_2 \Rightarrow (f^{-1}(B_1), f^{-1}(B_2))^C \in \tau_1 \hat{\times} \tau_2 \Rightarrow (f^{-1}(B_1), f^{-1}(B_2)) \in (\tau_1 \hat{\times} \tau_2)^C \Rightarrow$ $f^{-1}(\underline{B}) \in (\tau_1 \hat{\times} \tau_2)^C$.

(2) $\longrightarrow$ (3): Let $\underline{A} \in$ D (X). Since $f(\underline{A}) \subseteq C_{12}(f(\underline{A}))$. Then $f^{-1} f(\underline{A}) \subseteq f^{-1}[C_{12}(f(\underline{A}))] \Rightarrow \underline{A} \subseteq f^{-1}[C_{12}(f(\underline{A}))] \Rightarrow C_{12}(\underline{A}) \subseteq f^{-1}[C_{12}(f(\underline{A}))] \Rightarrow$ $f(C_{12}(\underline{A})) \subseteq C_{12}(f(\underline{A}))$(by (2))

(3) $\longrightarrow$(4): Let $\underline{B} \in$ D (Y). Take $\underline{A} = f^{-1}(\underline{B})$ using (3), we have $f[C_{12}(f^{-1}(\underline{B}))] \subseteq C_{12}(f\, f^{-1}(\underline{B})) \subseteq C_{12}(\underline{B}) \Rightarrow$ $C_{12}(f^{-1}(\underline{B})) \subseteq f^{-1}(C_{12}(\underline{B}))$

(4) $\longrightarrow$(1): Let $G_1 \in \theta_1$. Then $(G_1, \mathrm{Y}) \in \theta_1 \hat{\times} \theta_2$. Also, $(\varphi, G_1^{\,C}) \in (\theta_1 \hat{\times} \theta_2)^C$, using (4) we have: $C_{12}(f^{-1}(\varphi, G_1^{\,C})) \subseteq f^{-1}(C_{12}(\varphi, G_1^{\,C}))$ $= f^{-1}(\varphi, G_1^{\,C}) = (\varphi, f^{-1}(G_1^{\,C})) \subseteq C_{12}(\varphi, f^{-1}(G_1^{\,C}))$. So $f^{-1}(\varphi, G_1^{\,C}) \in (\tau_1 \hat{\times} \tau_2)^C$. Hence $(f^{-1}(G_1), \mathrm{X}) \in \tau_1 \hat{\times} \tau_2$.

Therefore $f^{-1}(G_1) \in \tau_1$ and the mapping $f:(X, \tau_1) \longrightarrow (\mathrm{Y}, \theta_1)$ is continuous. Similarly, we can show that $f:(X, \tau_2) \longrightarrow (\mathrm{Y}, \theta_2)$ is continuous. So $f:(X, \tau_1, \tau_2) \longrightarrow (\mathrm{Y}, \theta_1, \theta_2)$ is P- continuous. According to theorem 4.1, f is double continuous.

(1)$\longrightarrow$(5): Let f be double continuous. So, f : (X, τ_1, τ_2) $\longrightarrow$(Y, θ_1, θ_2) is P. continuous. Let $\underline{B} \in$ D (Y). Then $f^{-1}(I_{12}\,\underline{B}) = f^{-1}((B_1 \cap B_2^{\,02})^{01}, B_2^{\,02}) = (f^{-1}(B_1 \cap B_2^{\,02})^{01}, f^{-1} B_2^{\,02}) \subseteq ((f^{-1}(B_1 \cap B_2^{\,02}))^{01}, (f^{-1} B_2)^{02}) =$ $([f^{-1}(B_1) \cap f^{-1}(B_2)^{02}]^{01}, (f^{-1} B_2)^{02}) =$ $I_{12}\, f^{-1}(\underline{B})$.

(5) $\longrightarrow$ (1): Let $(B_1, \mathrm{Y}) \in \theta_1 \hat{\times} \theta_2$ $\Rightarrow f^{-1}(I_{12}(B_1, \mathrm{Y})) \subseteq I_{12}\, f^{-1}((B_1, \mathrm{Y})) \Rightarrow$ $f^{-1}(B_1, \mathrm{Y}) \subseteq I_{12}\, f^{-1}((B_1, \mathrm{Y})) \subseteq f^{-1}(B_1, \mathrm{Y})$. Then $f^{-1}(B_1, \mathrm{Y}) =$ $I_{12}\, f^{-1}((B_1, \mathrm{Y}))$ and therefore $f^{-1}(B_1, \mathrm{Y}) \in \tau_1 \hat{\times} \tau_2$. Hence $f^{-1} B_1 \in \tau_1$ and the mapping $f:(X, \tau_1) \longrightarrow (\mathrm{Y}, \theta_1)$ is continuous. Similarly if $B_2 \in \theta_2$, then $(\varphi, B_2) \in \theta_1 \hat{\times} \theta_2$ and apply the condition, we have $f^{-1}(\varphi, B_2) \in \tau_1 \hat{\times} \tau_2$ Hence $f^{-1} B_2 \in \tau_2$ and the mapping $f:(X, \tau_2) \longrightarrow (\mathrm{Y}, \theta_2)$ is continuous.

So, f : (X, τ_1, τ_2) $\longrightarrow$(Y, θ_1, θ_2) is P. continuous. So, according to theorem 4.1, f is double continuous.

By a similar way as in theorems 4.1, 4.2 we have the following theorems

Theorem 4.3. i) A mapping $f:(X,\tau_1,\tau_2)\rightarrow$(Y, θ_1,θ_2) is P- open iff

$f:(X,\tau_1\hat{\times}\tau_2)\rightarrow$(Y,$\theta_1\hat{\times}\theta_2$) is D open.

ii) A mapping $f:(X,\tau_1\hat{\times}\tau_2)\rightarrow$(Y,$\theta_1\hat{\times}\theta_2$) is D- open iff

$f[I_{12}(\underline{H})]\subseteq I_{12}[f(\underline{H})],\forall\ \underline{H}\in$ D (X).

Theorem 4.4. A surjection mapping $f:(X,\tau_1\hat{\times}\tau_2)\rightarrow$(Y,$\theta_1\hat{\times}\theta_2$) is D- open and D-continuous iff $f[I_{12}(\underline{H})]=I_{12}[f(\underline{H})],\ \forall\ \underline{H}\in$ D (X)

Theorem 4.5. i) A mapping $f:(X,\tau_1,\tau_2)\rightarrow$(Y, θ_1,θ_2) is P- closed iff

$f:(X,\tau_1\hat{\times}\tau_2)\rightarrow$(Y,$\theta_1\hat{\times}\theta_2$) is D-closed.

ii) A mapping $f:(X,\tau_1\hat{\times}\tau_2)\rightarrow$(Y,$\theta_1\hat{\times}\theta_2$) is D- closed iff

$C_{12}[f(\underline{H})]\subseteq f[C_{12}(\underline{H})],\forall\ \underline{H}\in$ D(X).

Theorem 4.6. A mapping $f:(X,\tau_1\hat{\times}\tau_2)\rightarrow$(Y,$\theta_1\hat{\times}\theta_2$) is a D- closed and D-continuous iff $C_{12}[f(\underline{H})]=f[C_{12}(\underline{H})],\forall\ \underline{H}\in$ D(X).

Corollary 4.7. Let $(X,\tau_1\hat{\times}\tau_2)$ and (Y,$\theta_1\hat{\times}\theta_2$) be double topological spaces. Then $f:(X,\tau_1\hat{\times}\tau_2)\rightarrow$(Y,$\theta_1\hat{\times}\theta_2$) is a double homeomorphism iff

$f:(X,\tau_1,\tau_2)\rightarrow$ (Y, θ_1,θ_2) is P-homeomorphism.

5. The relations between P*-continuous (resp P*-open, P*-closed)mappings and supra double continuous (rep supra double open, supra double closed) mappings:

In this section, we shall study the relation between P*-continuous (resp P*-open ,P*- closed) mappings and supra- double topological spaces. The proofs of the following results are similar to the proof of the results in section 4. So, we prove theorem 5.2 as an example, and we shall omitte the proof of the others.

Theorem 5.1. Let (X, τ_1,τ_2) and (Y, θ_1,θ_2) be bitopological spaces. Also, let (X,τ^*) and (Y,θ^*) be their associated supra-topological spaces. The following equivalent:

1) f : (X, τ_1,τ_2) $\rightarrow$ (Y, θ_1,θ_2) is P*-continuous .

2) f : (X, $\tau^*\hat{\times}\tau^*$) $\rightarrow$ (Y, $\theta^*\hat{\times}\theta^*$) is SD-continuous.

3) $f^{-1}(\underline{B})\in(\tau^*\hat{\times}\tau^*)^C$ $\forall\ \underline{B}\in(\theta^*\hat{\times}\theta^*)^C$

4) $f(C^*(\underline{A}))\subseteq C^*(f(\underline{A}))$ $\forall\ \underline{A}\in$ D (X)

5) $C^*[f^{-1}(\underline{B})]\subseteq f^{-1}(C^*(\underline{B}))$ $\forall\ \underline{B}\in$ D (Y)

6) $f^{-1}(I^*(\underline{B}))\subseteq I^*(f^{-1}(\underline{B}))$ $\forall\ \underline{B}\in$ D (Y)

Theorem 5.2. Let (X, τ_1,τ_2) and (Y,θ_1,θ_2) be bitopological spaces. Then the following are equivalent:

1) f : (X, τ_1,τ_2) $\rightarrow$(Y,θ_1,θ_2) is P* open.

2) $f:(X,\tau^*\hat{\times}\tau^*)\rightarrow$(Y, $\theta^*\hat{\times}\theta^*$) is a SD-open.

3) $f[I^*\underline{H}]\subseteq I^*[f(\underline{H})],\forall\ \underline{H}\in$ D (X).

Proof: $1\rightarrow 2$ Let $\underline{A}=(A_1,A_2)\in\tau^*\hat{\times}\tau^*$. Then $f(A_i)\in\theta^*$(i=1,2). So,

$f(\underline{A})=(f(A_1),f(A_2))\in\theta^*\hat{\times}\theta^*$. Hence $f:(X,\tau^*\hat{\times}\tau^*)\rightarrow$(Y, $\theta^*\hat{\times}\theta^*$) is a SD-open function.

$2\rightarrow 3$ Since, $I^*\underline{H}\subseteq\underline{H}$ $\forall\ \underline{H}\in$ D (X). Then $f[I^*\underline{H}]\subseteq f(\underline{H})\Rightarrow I^*\ f[I^*\underline{H}]=f[I^*\underline{H}]\subseteq I^*\ f(\underline{H})$. Hence $f[I^*\underline{H}]\subseteq I^*[f(\underline{H})],\forall\ \underline{H}\in$ D (X).

$3\rightarrow 1$ Straightforward.

Theorem 5.3. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be bitopological spaces. A mapping $f:(X, \tau^* \hat{\times} \tau^*) \to$ (Y, $\theta^* \hat{\times} \theta^*$) is SD- open and SD-continuous iff

$f[I^*\underline{H}] = I^*[f(\underline{H})]$ $\forall$ $\underline{H} \in$ D (X)

Theorem 5.4 Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be bitopological spaces. Then the following are equivalent:

i) $f:(X, \tau^* \hat{\times} \tau^*) \to$ (Y, $\theta^* \hat{\times} \theta^*$) is a SD-closed.

ii) f : (X, τ_1, τ_2) $\to$ (Y, θ_1, θ_2) is P^* closed.

iii) $C^*(f(\underline{H})) \subseteq f[C^*(\underline{H})]$, $\forall$ $\underline{H} \in$ D(X)

Theorem 5.5. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be bitopological spaces. A mapping

$f:(X, \tau^* \hat{\times} \tau^*) \to$ (Y, $\theta^* \hat{\times} \theta^*$) is SD- closed and SD- continuous iff

$C^*(f(\underline{H})) = f[C^*(\underline{H})]$, $\forall$ $\underline{H} \in$ D(X)

Corollary 5.6. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be bitopological spaces. Then

$f:(X, \tau^* \hat{\times} \tau^*) \to$ (Y, $\theta^* \hat{\times} \theta^*$) is a SD-homeomorphism iff

f : (X, τ_1, τ_2) $\to$ (Y, θ_1, θ_2) is P^* - homeomorphism function

6. Relation between D continuous (open, closed) mappings and SD continuous (open, closed) mappings:

Theorem 6.1. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be bitopological spaces and let,

f : (X, τ_1, τ_2) $\to$ (Y, θ_1, θ_2) be P- continuous (resp P- open). Then

$f:(X, \tau^* \hat{\times} \tau^*) \to$ (Y, $\theta^* \hat{\times} \theta^*$) is SD-continuous (resp SD-open).

Proof: It follows from the definition of P-continuous (resp P- open) and SD-continuous (resp SD-open).

Theorem 6.2. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be bitopological spaces and let

f : (X, τ_1, τ_2) $\to$ (Y, θ_1, θ_2) be P- closed and injection. Then

$f:(X, \tau^* \hat{\times} \tau^*) \to$ (Y, $\theta^* \hat{\times} \theta^*$) is SD-closed.

Proof: Let f : (X, τ_1, τ_2) $\to$ (Y, θ_1, θ_2) be a P- closed and injection. Let $\underline{H} \in \tau^{*C} \hat{\times} \tau^{*C}$. Then $\underline{H} = (K_1 \cap K_2, G_1 \cap G_2)$ such that $K_i, G_i \in \tau_i^C$ (i=1, 2). Then $f(\underline{H}) = f(K_1 \cap K_2, G_1 \cap G_2) = (f(K_1 \cap K_2), f(G_1 \cap G_2)) \subseteq$ $(f(K_1) \cap f(K_2), f(G_1) \cap f(G_2))$. $f(K_i), f(G_i) \in \theta_i^C \Rightarrow$

$f(\underline{H}) \in \theta^{*C} \hat{\times} \theta^{*C}$. Hence $f:(X, \tau^* \hat{\times} \tau^*)$ $\to$ (Y, $\theta^* \hat{\times} \theta^*$) is SD-closed.

Note that the mapping may be SD-continuous (SD-open and SD-closed) but not double continuous (double open and double closed) mapping as shown in the following example:

Example 6.3. Let X={a, b, c}. τ_1={ φ, X, {a}, {b, c}}, τ_2={ φ, X, {c}, {a, b}}. Also, let Y= {p, q, r} θ_1={ φ, Y, {r}, {p, q}}, θ_2={ φ, Y, {p}, {q, r}}. Then

$\tau_1 \hat{\times} \tau_2$ = {$\underline{\varphi}$, $\underline{X}$, (φ,{c}), (φ, {a, b}), (φ, X), ({a}, {a, b}), ({a}, X), ({b, c}, X)}

$\theta_1 \hat{\times} \theta_2$ = { $\underline{\varphi}$, $\underline{Y}$, (φ,{p}), (φ, {q, r}), (φ, Y), ({r}, {q, r}), ({r}, Y), ({p, q}, Y)}

let f : (X, τ_1, τ_2) $\to$ (Y, θ_1, θ_2) such that f (a) = p, f (b)= q, f (c)=r. Then f is not D-open since

(φ ,{c}) $\in$ $\tau_1 \hat{\times} \tau_2$ $\Rightarrow$ f (φ ,{c}) = (φ,{r})$\notin$ $\theta_1 \hat{\times} \theta_2$

let τ^* = { φ, X, {a}, {b, c}, {c}, {a, b},{a, c}}. Then τ^* is not a topology since {b, c} $\cap$ {a, b}= {b}$\notin$ τ^*. Also,

θ^* = { φ, Y, {r}, {p, q}, {p}, {q, r}, {r, p}} θ^* is not a topology since {p, q} $\cap$ {q, r}= {q}$\notin$ θ^*. Then

$\tau^* \hat{\times} \tau^*$={$\underline{\varphi}$, ($\varphi$, {a}), ($\varphi$, {b, c}),($\varphi$, {c}), ($\varphi$, {a, b}), ($\varphi$, {a, c}), ($\varphi$,X), ({a}, {a}), ({a}, {a, b}), ({a}, {a, c}), ({a}, X), ({c}, {c}), ({c}, {b, c}), ({c}, {a, c}), ({c}, X), ({b, c}, {b, c}), ({b, c}, X), ({a, b}, {a, b}), ({a, b}, X), ({a, c}, {a, c}), ({a, c}, X), $\underline{X}$ } and

$\theta^* \hat{\times} \theta^*$={ $\underline{\varphi}$, (φ, {r}), (φ, {p}), (φ, {p, q}), (φ, {q, r}), (φ, {r, p}), (φ, Y), ({r}, {r}), ({r}, {r, q}), ({r}, {r, p}), ({r}, Y), ({p}, {p}), ({p}, {p, q}), ({p}, {r, p}), ({p}, Y), ({p, q}, {p, q}), ({p, q}, Y), ({q, r}, {q, r}), ({q, r}, Y), ({r, p}, {r, p}), ({r, p}, Y), $\underline{Y}$ }.

Let $f:(X, \tau^* \hat{\times} \tau^*) \to$(Y, $\theta^* \hat{\times} \theta^*$). Then f is SD-continuous but

f : (X, $\tau_1 \hat{\times} \tau_2$) $\to$ (Y, $\theta_1 \hat{\times} \theta_2$) is not D-continuous, since

(φ ,{p}) $\in$ $\theta_1 \hat{\times} \theta_2$ but f^{-1} (φ ,{p}) = (φ,{a})$\notin$ $\tau_1 \hat{\times} \tau_2$.

Let $f:(X, \tau^* \hat{\times} \tau^*) \to$(Y, $\theta^* \hat{\times} \theta^*$). Then f is SD-open but

f : (X, $\tau_1 \hat{\times} \tau_2$) $\to$ (Y, $\theta_1 \hat{\times} \theta_2$) is not D- open, since

(φ ,{c}) $\in$ $\tau_1 \hat{\times} \tau_2$ but f (φ ,{c}) = (φ,{r})$\notin$ $\theta_1 \hat{\times} \theta_2$. Finally,

Let $f:(X, \tau^* \hat{\times} \tau^*) \to$(Y, $\theta^* \hat{\times} \theta^*$). Then f is SD-closed but

f : (X, $\tau_1 \hat{\times} \tau_2$) $\to$ (Y, $\theta_1 \hat{\times} \theta_2$) is not D-closed, since

({a, b}, X)$\in$ $(\tau_1 \hat{\times} \tau_2)^C$ but f ({a, b}, X) = ({p, q}, Y)$\notin$ $(\theta_1 \hat{\times} \theta_2)^C$.

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