The variational iteration method for exact solutions of fuzzy heat-like equations with variable coefficients

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Abstract. In this paper, the variational iteration method (VIM) and Buckley-Feuring method (BFM) are applied to find the exact fuzzy solution of the fuzzy heat-like equations in one and two dimensions with variable coefficients. Further a comparison between VIM-BFM and Seikkala solutions is provided.

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1. Introduction

Recently heat-like models for physical problems have been caught much attention. These models can exactly describe some nonlinear phenomena, for example, the most celebrated Navier-Stokes equations can be converted into various heat-like equation in some special cases. We suppose the existence of imprecise parameters in heat-like equations with variable coefficients. Since fuzzy sets theory is a powerful tool for modeling imprecision and for processing vagueness in mathematical models (Buckly 1999, Buckly 2000, Chalco-Cano2008, Nieto 2006), therefore, the purpose of this paper is using VIM and the same strategy as in Buckley and Feuring (Buckly 1999) for solving heat-like equations with fuzzy parameters.

The VIM (Abdou 2005, Shou 2008, Wazwaz 2004, Biazar 2007, Sadighi 2007, Abbasbandy 2009-64a, Abbasbandy 2009) gives rapid convergent successive approximations of the exact solution if such a solution exists without any restrictive assumption or transformation that may change the physical behavior of the problem. Very recently Allahviranloo et. al. (Allahviranloo 2009) discussed on the first order fuzzy differential equations by VIM. In this paper, we consider the iterative method for fuzzy heat-like equations in one and two dimensions with variable coefficients by VIM.

The paper is organized as follows: in Section 2, we call some fundamental results on fuzzy numbers. In Section 3 and 4, fuzzy heat-like equations and the VIM are illustrated, respectively. In Section 5, the same strategy as in Buckley-Feuring is presented for two-dimensional fuzzy heat-like equation. Some examples in Section 6 illustrated and finally conclusions are given in Section 7.

2. Preliminaries

We place a bar over a capital letter to denote a fuzzy subset of R^n. So, $\bar{X}$, $\bar{K}$, $\bar{C}$, etc. all represent fuzzy subsets of R^n for some n. We write $\mu_{\bar{A}}(t)$, a number in $[0,1]$, for the membership function of $\bar{A}$ evaluated at $t \in \mathrm{R}^n$. Define $\bar{A} \le \bar{B}$ when $\mu_{\bar{A}}(t) \le \mu_{\bar{B}}(t)$ for all t . An γ -cut of $\bar{A}$ is always a closed and bounded interval that written $\bar{A}[\gamma]$, is defined as $\{t \mid \mu_{\bar{A}}(t) \ge \gamma\}$, for $0 < \gamma \le 1$. We separately specify $\bar{A}[0]$ as the closure of the union of all the $\bar{A}[\gamma]$ for $0 < \gamma \le 1$. Let E shows a set of fuzzy numbers (Ma 1999).

We represent an arbitrary fuzzy number by an ordered pair of functions $\bar{A}[\gamma] = [A_1(\gamma), A_2(\gamma)]$, $0 \le \gamma \le 1$ which satisfy the following requirements

(a) $A_1(\gamma)$ is a bounded left continuous nondecreasing function over $[0,1]$,

(b) $A_2(\gamma)$ is a bounded left continuous non-increasing function over $[0,1]$,

(c) $A_1(\gamma) \le A_2(\gamma), 0 \le \gamma \le 1$.

A fuzzy set $\bar{A} = (a_1, a_2, a_3)$, $(a_1 < a_2 < a_3)$ is called triangular fuzzy number with peak (or center) a_2, left width $a_2 - a_1 > 0$ and right width $a_3 - a_2 > 0$, if its membership function has the following form

$$\mu_{\bar{A}}(t) = \begin{cases} 1 - \dfrac{(a_2 - t)}{a_2 - a_1}, & a_1 \le t \le a_2, \\ 1 - \dfrac{(t - a_2)}{a_3 - a_2}, & a_2 \le t \le a_3, \\ 0, & otherwise. \end{cases}$$

The support of $\bar{A}$ is $[a_1, a_3]$. We will write: (1) $\bar{A} > 0$ if $a_1 > 0$, (2) $\bar{A} \geq 0$ if $a_1 \geq 0$, (3) $\bar{A} < 0$ if $a_3 < 0$; and (4) $\bar{A} \leq 0$ if $a_3 \leq 0$. We adopt the general definition of a fuzzy number given in (Goetschel 1986).

3. Fuzzy heat-like equations

In this section, we consider the heat-like equations in one and two dimensions which can be written in the forms

(a) One-dimensional:

$$U_t(t,x) + p(x)U_{xx}(t,x) = F(t,x,k), \qquad (1)$$

(b) Two-dimensional:

$$U_t(t,x,y) + p(x)U_{xx}(t,x,y) + q(y)U_{yy}(t,x,y) = F(t,x,y,k), \qquad (2)$$

or

$$U_t(t,x,y) + q(y)U_{xx}(t,x,y) + p(x)U_{yy}(t,x,y) = F(t,x,y,k), \qquad (3)$$

subject to certain initial and boundary conditions.

These initial and boundary conditions, in state two-dimensional, can come in a variety of forms such as $U(0,x,y) = c_1$ or $U(0,x,y) = g_1(x,y,c_2)$ or $U(M_1,x,y) = g_2(x,y,c_3,c_4)$,....

In this paper the method is applied for the heat-like equation (2). For Eqs. (1) and (3), it is similar to (2), so we will omit them. In following lines, components of Eq. (2) are enumerated:

- $I_j = [0, M_j]$ are three intervals, which $M_j > 0 (j = 1,2,3)$.
- $F(t,x,y,k)$, $U(t,x,y)$, $p(x)$ and $q(y)$ will be continuous functions for $(t,x,y) \in \prod_{j=1}^{3} I_j$.
- $p(x)$ and $q(y)$ have a finite number of roots for each $(x,y) \in I_2 \times I_3$.
- $k = (k_1, \mathrm{K}, k_n)$ and $c = (c_1, \mathrm{K}, c_m)$ are vectors of constants with k_j in interval J_j and c_r in interval L_r.

Assume the Eq. (2) has a solution

$$U(t,x,y) = G(t,x,y,k,c), \qquad (4)$$

for continuous G ($G_t(t,x,y,k,c) + p(x)G_{xx}(t,x,y,k,c) + q(y)G_{yy}(t,x,y,k,c)$ is continuous for $(t,x,y) \in \prod_{j=1}^{3} I_j, k \in J, c \in L$) with $(t,x,y) \in \prod_{j=1}^{3} I_j$, $k \in J = \prod_{j=1}^{n} J_j$ and $c \in L = \prod_{r=1}^{m} L_r$.

Now suppose the value of the k_j and c_r are imprecise. We will model this uncertainty by substitute triangular fuzzy numbers for the k_j and c_r. If we fuzzify Eq. (2), then we obtain the fuzzy heat-like equation. Using the extension principle we compute $\bar{F}$ from F where $\bar{F}(t,x,y,\bar{K})$ has $\bar{K} = (\bar{K}_1, \mathrm{K}, \bar{K}_n)$ for $\bar{K}_j$ a triangular fuzzy number in J_j, $1 \leq j \leq n$. The function U become $\bar{U}$, where $\bar{U}: \prod_{j=1}^{3} I_j \to E$. That is, $\bar{U}(t,x,y)$ is a fuzzy number. The fuzzy heat-like equation is

$$\bar{U}_t + p(x)\bar{U}_{xx} + q(y)\bar{U}_{yy} = \bar{F}(t,x,y,\bar{K}), \qquad (5)$$

subject to certain initial and boundary conditions. The initial and boundary conditions can be of the form $\bar{U}(0,x,y) = \bar{C}_1$ or $\bar{U}(0,x,y) = \bar{g}_1(x,y,\bar{C}_2)$ or $\bar{U}(M_1,x,y) = \bar{g}_2(x,y,\bar{C}_3,\bar{C}_4)$,.... The $\bar{g}_j$ is the extension principle of g_j. We wish to solve the problem given in Eq. (5). Finally, we fuzzify G in Eq. (4). Let $\bar{Z}(t,x,y) = \bar{G}(t,x,y,\bar{K},\bar{C})$ where $\bar{Z}$ is computed using the extension principle and is a fuzzy solution. In Section 5, we will discuss solution with the same strategy as Buckley-Feuring for fuzzy heat-like equation. Let $\bar{K}[\gamma] = \prod_{j=1}^{n} \bar{K}_j[\gamma]$ and $\bar{C}[\gamma] = \prod_{r=1}^{m} \bar{C}_r[\gamma]$.

4. The variational iteration method

To illustrate the basic idea of the VIM we consider the following model PDE

$$L_t U + L_x U + L_y U + NU = F(t,x,y,k), \quad (6)$$

where L_t, L_x and L_y are linear operators of t, x and y, respectively, and N is a nonlinear operator, also $F(t,x,y,k)$ is the source non-homogeneous term. According to the VIM, we can express the following correction functional in t-direction as follows

$$U_{n+1}(t,x,y) = U_n(t,x,y) + \int_0^t \lambda\{L_s U_n + (L_x + L_y + N)\tilde{U}_n - F\} ds, \qquad (7)$$

where λ is general Lagrange multiplier (He 2004), which can be identified optimally via the variational theory (He 2006, Wazwaz 2007), and $\tilde{U}_n$ is a restricted variation which means $\delta\tilde{U}_n = 0$. By this method, we determine first the Lagrange multiplier λ which will be identified optimally. The successive approximations U_{n+1}, $n \geq 0$, of the solution U will be readily obtained by suitable choice of trial function U_0. Consequently, the solution is given as

$$U(t,x,y) = \lim_{n \to \infty} U_n(t,x,y). \qquad (8)$$

According to the VIM, we construct a correction functional for Eq. (2) in the form

$$U_{n+1}(t,x,y) = U_n(t,x,y) + \int_0^t \lambda(s) \{(U_n)_s + p(x)(\tilde{U}_n)_{xx} + q(y)(\tilde{U}_n)_{yy} - F\}ds, \quad (9)$$

where $n \geq 0$ and λ is a Lagrange multiplier. Making Eq. (9) stationary with respect to U_n, we have

$$\lambda'(s) = 0,$$
$$1 + \lambda(s)|_{s=t} = 0,$$

hence,the Lagrange multiplier is $\lambda = -1$. Submitting the results into Eq. (9) leads to the following iteration formula

$$U_{n+1}(t,x,y) = U_n(t,x,y) - \int_0^t \{(U_n)_s + p(x)(U_n)_{xx} + q(y)(U_n)_{yy} - F\}ds. \quad (10)$$

Iteration formula start with an initial approximation, for example $U_0(t,x,y) = U(0,x,y)$. Also the VIM used for system of linear and nonlinear partial differential equations (Wazwaz 2007) which handled in obtain Seikkala solution.

5. Buckley-Feuring Solution (BFS) and Seikkala Solution (SS)

In (Buckly 1990), Buckley-Feuring present the BFS. For all t, x, y and γ,

$$\tilde{Z}(t,x,y)[\gamma] = [z_1(t,x,y,\gamma), z_2(t,x,y,\gamma)], \quad (11)$$

and

$$\tilde{F}(t,x,y,\tilde{K})[\gamma] = [F_1(t,x,y,\gamma), F_2(t,x,y,\gamma)], \quad (12)$$

that by definition

$$z_1(t,x,y,\gamma) = \min\{G(t,x,y,k,c) \mid k \in \tilde{K}[\gamma], c \in \tilde{C}[\gamma]\}, \quad (13)$$

$$z_2(t,x,y,\gamma) = \max\{G(t,x,y,k,c) \mid k \in \tilde{K}[\gamma], c \in \tilde{C}[\gamma]\}, \quad (14)$$

and

$$F_1(t,x,y,\gamma) = \min\{F(t,x,y,k) \mid k \in \tilde{K}[\gamma]\}, \quad (15)$$

$$F_2(t,x,y,\gamma) = \max\{F(t,x,y,k) \mid k \in \tilde{K}[\gamma]\}. \quad (16)$$

Assume that $p(x) > 0$, $q(y) > 0$ and the $z_i(t,x,y,\gamma)$, $i = 1,2$, have continuous partial so that $(z_i)_t + p(x)(z_i)_{xx} + q(y)(z_i)_{yy}$ is continuous for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all γ. Define

$$\Gamma(t,x,y,\gamma) = (z_1)_t + p(x)(z_1)_{xx} + q(y)(z_1)_{yy}, \; (z_2)_t + p(x)(z_2)_{xx} + q(y)(z_2)_{yy}], \quad (17)$$

for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all γ. If, for each fixed $(t,x,y) \in \prod_{j=1}^3 I_j$, $\Gamma(t,x,y,\gamma)$ defines the γ-cut of a fuzzy number, then will be said that $\tilde{Z}(t,x,y)$ is differentiable and is written

$$\tilde{Z}_t[\gamma] + p(x)\tilde{Z}_{xx}[\gamma] + q(y)\tilde{Z}_{yy}[\gamma] = \Gamma(t,x,y,\gamma), \quad (18)$$

for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all γ.

Sufficient conditions for $\Gamma(t,x,y,\gamma)$ to define γ-cuts of a fuzzy number are [14]:

(i) $(z_1(t,x,y,\gamma))_t + p(x)(z_1(t,x,y,\gamma))_{xx} + q(y)(z_1(t,x,y,\gamma))_{yy}$ is an increasing function of γ for each $(t,x,y) \in \prod_{j=1}^3 I_j$;

(ii) $(z_2(t,x,y,\gamma))_t + p(x)(z_2(t,x,y,\gamma))_{xx} + q(y)(z_2(t,x,y,\gamma))_{yy}$ is a decreasing function of γ for each $(t,x,y) \in \prod_{j=1}^3 I_j$; and

(iii) $(z_1(t,x,y,1))_t + p(x)(z_1(t,x,y,1))_{xx} + q(y)(z_1(t,x,y,1))_{yy} \leq (z_2(t,x,y,1))_t + p(x)(z_2(t,x,y,1))_{xx} + q(y)(z_2(t,x,y,1))_{yy}$ for $(t,x,y) \in \prod_{j=1}^3 I_j$.

Now can suppose that the $z_i(t,x,y,\gamma)$ have continuous partial so

$$(z_i)_t + p(x)(z_i)_{xx} + q(y)(z_i)_{yy},$$

is continuous on $\prod_{j=1}^3 I_j \times [0,1]$, $i = 1,2$. Hence, if conditions (i)-(iii) above hold, $\tilde{Z}(t,x,y)$ is differentiable.

For $\tilde{Z}(t,x,y)$ to be a BFS of the fuzzy heat-like equation we need: (a) $\tilde{Z}(t,x,y)$ differentiable; (b) Eq. (5) holds for $\tilde{U}(t,x,y) = \tilde{Z}(t,x,y)$; and (c) $\tilde{Z}(t,x,y)$ satisfies the initial and boundary conditions. Since no exist specified any particular initial and boundary conditions then only is checked if Eq. (5) holds.

$\tilde{Z}(t,x,y)$ is a BFS (without the initial and boundary conditions) if $\tilde{Z}(t,x,y)$ is differentiable and

$$\tilde{Z}_t + p(x)\tilde{Z}_{xx} + q(y)\tilde{Z}_{yy} = \tilde{F}(t,x,y,\tilde{K}), \quad (19)$$

or the following equations must hold

$$(z_1)_t + p(x)(z_1)_{xx} + q(y)(z_1)_{yy} = F_1(t,x,y,\gamma), \quad (20)$$

$$(z_2)_t + p(x)(z_2)_{xx} + q(y)(z_2)_{yy} = F_2(t,x,y,\gamma), \quad (21)$$

for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all γ.

Now we will present a sufficient condition for the BFS to exist such as Buckley and Feuring.

Since there are such a variety of possible initial and boundary conditions, hence we will omit them from the following Theorem. One must separately check out the initial and boundary conditions. So, we will omit the constants $c_r, 1 \le r \le m$, from the problem. Therefore, Eq. (4) becomes $U(t,x,y) = G(t,x,y,k)$, so $\tilde{Z}(t,x,y) = \tilde{G}(t,x,y,\tilde{K})$.

Theorem 1. Suppose $\tilde{Z}(t,x,y)$ is differentiable.

(a) If

$$p(x) > 0, q(y) > 0, (x,y) \in I_2 \times I_3, \qquad (22)$$

and

$$\frac{\partial G}{\partial k_j}\frac{\partial F}{\partial k_j} > 0, \qquad (23)$$

for $j = 1,\ldots,n$, Then BFS= $\tilde{Z}(t,x,y)$.

(b) If relations (22) does not hold or relation (23) does not hold for some j, then $\tilde{Z}(t,x,y)$ is not a BFS.

Proof. It is similar to proof of Theorem 1 in (Buckly 1999).

Therefore, if $\tilde{Z}(t,x,y)$ is a BFS and it satisfies the initial and boundary conditions we will say that $\tilde{Z}(t,x,y)$ is a BFS satisfying the initial and boundary conditions. If $\tilde{Z}(t,x,y)$ is not a BFS, then we will consider the SS. Now let us define the SS (Seikkala 1987). Let

$$\tilde{U}(t,x,y)[\gamma] = [u_1(t,x,y,\gamma), u_2(t,x,y,\gamma)].$$

For example suppose $p(x) > 0$ and $q(y) < 0$, so consider the system of heat-like equations

$$(u_1)_t + p(x)(u_1)_{xx} + q(y)(u_2)_{yy} = F_1(t,x,y,\gamma), \qquad (24)$$

$$(u_2)_t + p(x)(u_2)_{xx} + q(y)(u_1)_{yy} = F_2(t,x,y,\gamma), \qquad (25)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and all $\gamma \in [0,1]$. We append to Eqs. (24) and (25) any initial and boundary conditions. For example, if it was $\tilde{U}(0,x,y) = \tilde{C}_1$ then we add

$$u_1(0,x,y,\gamma) = c_{11}(\gamma), \qquad (26)$$

$$u_2(0,x,y,\gamma) = c_{12}(\gamma), \qquad (27)$$

where $\tilde{C}_1[\gamma] = [c_{11}(\gamma), c_{12}(\gamma)]$. Let $u_i(t,x,y,\gamma), (i = 1,2)$ solve Eqs. (24) and (25), plus initial and boundary conditions. If

$$[u_1(t,x,y,\gamma), u_2(t,x,y,\gamma)], \qquad (28)$$

defines the γ-cut of a fuzzy number, for all $(t,x,y) \in \prod_{j=1}^{3} I_j$, then $\tilde{U}(t,x,y)$ is the SS.

We will say that derivative condition holds for fuzzy heat-like equation when Eqs. (22) and (23) are true.

Theorem 2. (1) If BFS= $\tilde{Z}(t,x,y)$, then SS= $\tilde{Z}(t,x,y)$. (2) If SS= $\tilde{U}(t,x,y)$ and the derivative condition holds, then BFS= $\tilde{U}(t,x,y)$.

Proof. (1) Follows from the definition of BFS and SS. (2) If SS= $\tilde{U}(t,x,y)$ then the Seikkala derivative (Buckly 2000) exists and since the derivative condition holds, therefore, Eqs. following holds

$$(u_1)_t + p(x)(u_1)_{xx} + q(y)(u_1)_{yy} = F_1(t,x,y,\gamma), \quad (29)$$

$$(u_2)_t + p(x)(u_2)_{xx} + q(y)(u_2)_{yy} = F_2(t,x,y,\gamma). \quad (30)$$

Also suppose one $k_j = k$ and $\frac{\partial G}{\partial k} < 0$, $\frac{\partial F}{\partial k} < 0$ (the other cases are similar and are omitted).

We see

$$z_1(t,x,y,\gamma) = G(t,x,y,k_2(\gamma)), \qquad (31)$$

$$z_2(t,x,y,\gamma) = G(t,x,y,k_1(\gamma)), \qquad (32)$$

$$F_1(t,x,y,\gamma) = F(t,x,y,k_2(\gamma)), \qquad (33)$$

$$F_2(t,x,y,\gamma) = F(t,x,y,k_1(\gamma)). \qquad (34)$$

Now look at Eqs. (20) and (21) also Eqs. (13) and (14), implies that

$$u_1(t,x,y,\gamma) = G(t,x,y,k_2(\gamma)) = z_1(t,x,y,\gamma),$$

$$u_2(t,x,y,\gamma) = G(t,x,y,k_1(\gamma)) = z_2(t,x,y,\gamma).$$

Therefore BFS= $\tilde{U}(t,x,y)$.

Remark 1. The Theorem 1 hold for Eq. (3) and the proof is similar to Theorem 1 in (Buckly 1999).

Lemma 1. Consider Eq. (1). Assume $\tilde{Z}(t,x)$ is differentiable.

(a) If

$$p(x) > 0, x \in I_2, \qquad (35)$$

and

$$\frac{\partial G}{\partial k_j}\frac{\partial F}{\partial k_j} > 0, \qquad (36)$$

for $j = 1,\ldots,n$, Then BFS= $\tilde{Z}(t,x)$.

(b) If relation (35) does not hold or relation (36) does not hold for some j, then $\tilde{Z}(t,x)$ is not a BFS.

Proof. It is similar to Theorem 1 in (Buckly 1999).

6. Examples

We consider the following illustrating examples.

Example 1. We first consider the one-dimensional initial value problem

$$U_t + \frac{1}{2}x^2 U_{xx} = k, \qquad (37)$$

subject to the initial condition $U(0,x) = cx^2$ and $t \in (0, M_1], x \in (0, M_2)$. Let $k \in [0, J]$ and $c \in [0, L]$ are constants. According to the VIM, a correct functional for Eq. (37) from Eq. (10) can be constructed as follows

$$U_{n+1}(t,x) = U_n(t,x) - \int_0^t \{(U_n(s,x))_s + \frac{1}{2}x^2 (U_n(s,x))_{xx} - k\}ds.$$

Beginning with an initial approximation $U_0(t,x) = U(0,x) = cx^2$, we can obtain the following successive approximations

$$U_1(t,x) = kt + cx^2(1-t),$$
$$U_2(t,x) = kt + cx^2(1-t+\frac{t^2}{2}),$$
$$U_3(t,x) = kt + cx^2(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}),$$

and

$$U_n(t,x) = kt + cx^2(1-t+\frac{t^2}{2!}+\Lambda+(-1)^n\frac{t^n}{n!}), n \geq 1.$$

The VIM admits the use of

$$U(t,x) = \lim_{n\to\infty} U_n(t,x),$$

which gives the exact solution

$$U(t,x) = kt + cx^2 e^{-t}.$$

Now we fuzzify $F(t,x,k) = k$ and $G(t,x,k,c) = kt + cx^2 e^{-t}$. Clearly $\tilde{F}(t,x,\tilde{K}) = \tilde{K}$ so that $F_1(t,x,\gamma) = k_1(\gamma)$ and $F_2(t,x,\gamma) = k_2(\gamma)$. Also $\tilde{G}(t,x,\tilde{K},\tilde{C}) = \tilde{K}t + \tilde{C}x^2 e^{-t}$, therefore,

$$z_i(t,x,\gamma) = k_i(\gamma)t + c_i(\gamma)x^2 e^{-t},$$

for $i = 1,2$, $\tilde{K}[\gamma] = [k_1(\gamma), k_2(\gamma)]$ and $\tilde{C}[\gamma] = [c_1(\gamma), c_2(\gamma)]$. $\tilde{Z}(t,x)$ is differentiable because $(z_i)_t + \frac{1}{2}x^2(z_i)_{xx} = k_i(\gamma), i = 1,2$. That is, $\tilde{Z}_t + \frac{1}{2}x^2\tilde{Z}_{xx} = \tilde{K}$, a fuzzy number. Since $p(x) > 0$, $\frac{\partial G}{\partial k} > 0$ and $\frac{\partial F}{\partial k} > 0$, Lemma (5) implies the result that $\tilde{Z}(t,x)$ is a BFS. We easily see that

$$z_i(0,x,\gamma) = c_i(\gamma)x^2,$$

for $i = 1,2$, so $\tilde{Z}(t,x)$ also satisfies the initial condition. The BFS that satisfies the initial condition may be written as

$$\tilde{Z}(t,x) = \tilde{K}t + \tilde{C}x^2 e^{-t},$$

for all $(t,x) \in [0, M_1] \times (0, M_2)$.

Example 2. Consider the two-dimensional heat-like equation with variable coefficients as

$$U_t(t,x,y) + \frac{1}{2}x^2 U_{xx}(t,x,y) + \frac{1}{2}y^2 U_{yy}(t,x,y) = kx^2 y,$$
$$U(0,x,y) = c_1 y^2 - c_2 x,$$

which $x, y \in (0,1)$, $t \in (0, M]$, $k \in [0, J]$ and $c_j \in [0, L_j], j = 1,2$.

Similarly we can establish an iteration formula in the form

$$U_{n+1} = U_n - \int_0^t \{(U_n)_s + \frac{1}{2}x^2(U_n)_{xx} + \frac{1}{2}y^2(U_n)_{yy} - kx^2 y\}ds. \quad (38)$$

We begin with an initial arbitrary approximation: $U_0(t,x,y) = U(0,x,y) = c_1 y^2 - c_2 x$, and using the iteration formula (38), we obtain the following successive approximations

$$U_1(t,x,y) = c_1 y^2(1-t) - c_2 x + kx^2 yt,$$
$$U_2(t,x,y) = c_1 y^2(1-t+\frac{t^2}{2!}) - kx^2 y(-t+\frac{t^2}{2!}) - c_2 x,$$
$$U_3(t,x,y) = c_1 y^2(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}) - kx^2 y(-t+\frac{t^2}{2!}-\frac{t^3}{3!}) - c_2 x,$$

and

$$U_n(t,x,y) = c_1 y^2(1-t+\Lambda+\frac{(-1)^n t^n}{n!}) - kx^2 y(-t+\Lambda+\frac{(-1)^n t^n}{n!}) - c_2 x, n \geq 1.$$

Then, the exact solution is given by

$$U(t,x,y) = G(t,x,y,k,c) = c_1 y^2 e^{-t} - kx^2 y(e^{-t}-1) - c_2 x.$$

Fuzzify F and G producing their γ-cuts

$$z_1(t,x,y,\gamma) = c_{11}(\gamma)y^2 e^{-t} - k_1(\gamma)x^2 y(e^{-t}-1) - c_{22}(\gamma)x,$$
$$z_2(t,x,y,\gamma) = c_{12}(\gamma)y^2 e^{-t} - k_2(\gamma)x^2 y(e^{-t}-1) - c_{21}(\gamma)x,$$
$$F_1(t,x,y,\gamma) = k_1(\gamma)x^2 y,$$
$$F_2(t,x,y,\gamma) = k_2(\gamma)x^2 y,$$

where $\tilde{K} = [k_1(\gamma), k_2(\gamma)]$ and $\tilde{C}_j[\gamma] = [c_{j1}(\gamma), c_{j2}(\gamma)], j = 1,2$. We first check to see if $\tilde{Z}(t,x,y)$ is differentiable. We compute

$$[(z_1)_t + \frac{1}{2}x^2(z_1)_{xx} + \frac{1}{2}y^2(z_1)_{yy}, (z_2)_t + \frac{1}{2}x^2(z_2)_{xx} + \frac{1}{2}y^2(z_2)_{yy}],$$

which are γ-cuts of $\tilde{K}x^2 y$ i.e. γ-cuts of a fuzzy number. Hence, $\tilde{Z}(t,x,y)$ is differentiable.

Since the partial F and G with respect to k, $p(x)$ and $q(y)$ are positive then Theorem (5) tells us that $\tilde{Z}(t,x,y)$ is a BFS. The initial condition is

$$z_1(0,x,y) = c_{11}(\gamma)y^2 - c_{22}(\gamma)x,$$
$$z_2(0,x,y) = c_{12}(\gamma)y^2 - c_{21}(\gamma)x,$$

which are true. Therefore, $\tilde{Z}(t,x,y)$ is a BFS which also satisfies the initial condition. This BFS may be written

$$\tilde{Z}(t,x,y) = \tilde{C}_1 y^2 e^{-t} - \tilde{K}x^2 y(e^{-t}-1) - \tilde{C}_2 x,$$

for all $x, y \in (0,1)$, $t \in [0,M]$. . We consider the one-dimensional heat-like model

$$U_t(t,x) + (\frac{1}{2} - x)U_{xx}(t,x) = -kx^2t^2, \quad U(0,x) = cx^2, \qquad (39)$$

which $t \in (0,1]$, $x \in (0,\frac{1}{2})$ and the value of parameters k and c are in intervals $[0,J]$ and $[0,L]$, respectively.

We can obtain the following iteration formula for the Eq. (39)

$$U_{n+1}(t,x) = U_n(t,x) - \int_0^t \{(U_n(s,x))_s + (\frac{1}{2} - x)(U_n(s,x))_{xx} + kx^2s^2\}ds. \qquad (40)$$

We begin with an initial approximation: $U_0(t,x) = cx^2$. By (40), after than two iterations the exact solution is given in the closed forms as

$$U(t,x) = G(t,x,k,c) = \frac{1}{12}kt^4 - \frac{1}{6}kxt^4 - \frac{1}{3}kx^2t^3 + cx^2 + 2cxt - ct.$$

Since $\frac{\partial F}{\partial k} = -x^2t^2 < 0$ and $\frac{\partial G}{\partial k} = \frac{1}{12}t^4 - \frac{1}{6}xt^4 - \frac{1}{3}x^2t^3 > 0$, for

$$0 < t \le 1 \quad and \quad 0 < x < \frac{1}{4}(-t + \sqrt{4t + t^2})$$

then there is no BFS (Lemma (5)). We proceed to look for a SS. We must solve

$$(u_1(t,x,\gamma))_t + (\frac{1}{2} - x)(u_1(t,x,\gamma))_{xx} = -k_2(\gamma)x^2t^2,$$
$$(u_2(t,x,\gamma))_t + (\frac{1}{2} - x)(u_2(t,x,\gamma))_{xx} = -k_1(\gamma)x^2t^2,$$

subject to

$$u_i(0,x,\gamma) = c_i(\gamma)x^2,$$

for $i = 1,2$, $\tilde{K}[\gamma] = [k_1(\gamma), k_2(\gamma)]$ and $\tilde{C}[\gamma] = [c_1(\gamma), c_2(\gamma)]$. By VIM, the solution is

$$u_1(t,x,\gamma) = \frac{1}{12}k_2(\gamma)t^4 - \frac{1}{6}k_2(\gamma)xt^4 - \frac{1}{3}k_2(\gamma)x^2t^3 + c_1(\gamma)x^2 + 2c_1(\gamma)xt - c_1(\gamma)t,$$
$$u_2(t,x,\gamma) = \frac{1}{12}k_1(\gamma)t^4 - \frac{1}{6}k_1(\gamma)xt^4 - \frac{1}{3}k_1(\gamma)x^2t^3 + c_2(\gamma)x^2 + 2c_2(\gamma)xt - c_2(\gamma)t. \qquad (41)$$

Now we denote

$$[u_1(t,x,\gamma), u_2(t,x,\gamma)],$$

defines γ-cuts of a fuzzy number on area as $\Re$. Since $u_i(t,x,\gamma)$ are continuous and $u_1(t,x,1) = u_2(t,x,1)$ then we only require to check if $\frac{\partial u_1}{\partial \gamma} > 0$ and $\frac{\partial u_2}{\partial \gamma} < 0$. Since $\tilde{K}$ and $\tilde{C}$ are triangular fuzzy numbers, hence, we pick simple fuzzy parameter so that $k_1'(\gamma) = c_1'(\gamma) = b > 0$ and $k_2'(\gamma) = c_2'(\gamma) = -b$. The 'prime' denotes differentiation with respect to γ. Then, for a SS we need

$$\frac{\partial u_1}{\partial \gamma} = -\frac{1}{12}bt^4 + \frac{1}{6}bxt^4 + \frac{1}{3}bx^2t^3 + bx^2 + 2bxt - bt = b(-\frac{1}{12}t^4 + \frac{1}{6}xt^4 + \frac{1}{3}x^2t^3 + x^2 + 2xt - t) > 0,$$
$$\frac{\partial u_2}{\partial \gamma} = \frac{1}{12}bt^4 - \frac{1}{6}bxt^4 - \frac{1}{3}bx^2t^3 - bx^2 - 2bxt + bt = -b(-\frac{1}{12}t^4 + \frac{1}{6}xt^4 + \frac{1}{3}x^2t^3 + x^2 + 2xt - t) < 0. \qquad (42)$$

Therefore inequalities (42) hold if

$$-\frac{1}{12}t^4 + \frac{1}{6}xt^4 + \frac{1}{3}x^2t^3 + x^2 + 2xt - t > 0, \qquad (43)$$

for $x \in (0,\frac{1}{2})$ and $t \in (0,1]$. The inequality (43) holds if we have

$$0 < t \le 1, \quad \frac{-12t - t^4 + \sqrt{144t + 144t^2 + 60t^4 + 24t^5 + 4t^7 + t^8}}{12 + 4t^3} < x < \frac{1}{2}.$$

We find that

$$\max\{\frac{-12t - t^4 + \sqrt{144t + 144t^2 + 60t^4 + 24t^5 + 4t^7 + t^8}}{12 + 4t^3} \mid 0 < t \le 1\} = 0.40103.$$

Hence we may choose $\Re$ by the above assumptions in form as

$$\Re = \{(t,x) \,|\, 0 < t \le 1 \& 0.401031 \le x < \frac{1}{2}\},$$

and the SS exists on $\Re$ in form Eqs. (41). . We consider the one-dimensional heat-like model,

$$U_t(t,x) - U_{xx}(t,x) = -k\cos x,$$
$$U(0,x) = c\sin x,$$

which $x \in (0, \frac{\pi}{2})$, $t \in (0, M]$ and the value of parameters k and c are in intervals $[0, J]$ and $[0, L]$, respectively.

We can obtain the following iteration formula

$$U_{n+1}(t,x) = U_n(t,x) - \int_0^t \{(U_n(s,x))_s - (U_n(s,x))_{xx} + k\cos x\}ds. \quad (44)$$

We begin with an initial approximation: $U_0(t,x) = U(0,x) = c\sin x$. By (44), the following successive approximation are obtained

$$U_1(t,x) = c\sin x(1-t) - kt\cos x,$$
$$U_2(t,x) = c\sin x(1 - t + \frac{t^2}{2!}) + k\cos x(-t + \frac{t^2}{2!}),$$

and

$$U_n(t,x) = c\sin x(1 - t + \Lambda + \frac{(-1)^n t^n}{n!}) + k\cos x(-t + \Lambda + \frac{(-1)^n t^n}{n!}), n \ge 1.$$

We, therefore, obtain

$$U(t,x) = G(t,x,k,c) = ce^{-t}\sin x + k\cos x(e^{-t} - 1),$$

which is the exact solution. There is no BFS because $p(x) = -1 < 0$ (Lemma (5)). We proceed to look for a SS. We must solve

$$(u_1(t,x,\gamma))_t - (u_2(t,x,\gamma))_{xx} = -k_2(\gamma)\cos x,$$
$$(u_2(t,x,\gamma))_t - (u_1(t,x,\gamma))_{xx} = -k_1(\gamma)\cos x,$$

subject to

$$u_i(0,x,\gamma) = c_i(\gamma)\sin x,$$

$i = 1,2$, $\tilde{K}[\gamma] = [k_1(\gamma), k_2(\gamma)]$ and $\tilde{C}[\gamma] = [c_1(\gamma), c_2(\gamma)]$. The solution is

$$u_1(t,x,\gamma) = c_1(\gamma)\cosh t\sin x - c_2(\gamma)\sinh t\sin x + k_1(\gamma)\cos x(\cosh t - 1) - k_2(\gamma)\cos x\sinh t,$$
$$u_2(t,x,\gamma) = c_2(\gamma)\cosh t\sin x - c_1(\gamma)\sinh t\sin x + k_2(\gamma)\cos x(\cosh t - 1) - k_1(\gamma)\cos x\sinh t.$$

We only need to check if $\frac{\partial u_1}{\partial\gamma} > 0$ and $\frac{\partial u_2}{\partial\gamma} < 0$, since the u_i are continuous and $u_1(t,x,1) = u_2(t,x,1)$. We pick simple fuzzy parameter so that $k_1'(\gamma) = c_1'(\gamma) = b > 0$ and $k_2'(\gamma) = c_2'(\gamma) = -b$. Then, for a SS we require

$$\frac{\partial u_1}{\partial\gamma} = b\sin x(\cosh t + \sinh t) + b\cos x(\cosh t - 1 + \sinh t) > 0,$$
$$\frac{\partial u_2}{\partial\gamma} = -b\sin x(\cosh t + \sinh t) - b\cos x(\cosh t - 1 + \sinh t) < 0. \quad (45)$$

Since (45) holds for each $t \in (0, M]$ and $x \in (0, \frac{\pi}{2})$, therefore, $\tilde{U}(t,x)$ is SS and

$$\tilde{U}(t,x) = \tilde{C}\cosh t\sin x - \tilde{C}\sinh t\sin x + \tilde{K}\cos x(\cosh t - 1) - \tilde{K}\sinh t\cos x,$$

for all $t \in [0, M]$ and $x \in (0, \frac{\pi}{2})$.

7. Conclusion

In this paper, by the VIM, we obtain the exact solutions of various kinds of fuzzy heat-like equations. The VIM produces the terms of a sequence using the iteration of the correction functional which converges to the exact solution rapidly. Application of this method is easy and calculation of successive approximations is direct and straightforward. We using the VIM and strategy based on (Buckly 1999) introduced two type of solutions, the Buckley-Feuring solution and the Seikkala solution. If the BFS fails to exist and when the SS fails to exist we offer no solution to the fuzzy heat-like equations.

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References

1. Buckley JJ, Feuring T. Introduction to fuzzy partial diferential equations. Fuzzy Sets and Systems 1999:105:241-248.
2. Buckley JJ, Feuring T. Fuzzy diferential equations. Fuzzy Sets and Systems 2000:110: 43-54.
3. Chalco-Cano Y, Roman-Flores Y. On new solutions of fuzzy differential equations. Chaos, Solitons & Fractals 2008:38:112-119.
4. Nieto JJ, Rodriguez-Lopez R. Bounded solutions for fuzzy differential and integral

equations. Chaos, Solitons & Fractals 2006:27 :1376-1386.
5. Abdou MA, Soliman AA. Variational iteration method for solving Burger's and coupled Burger's equation. Journal Comput. Appl. Math. 2005:181: 245-251.
6. Shou DJ, He JH. Beyond Adomian method: The variational iteration method for solving heat-like and wave-like equations with variable coefficients Phys. Let. A 2008:372: 233-237.
7. Wazwaz AM, Gorguis A. Exact solutions for heat-like and wave-like equations with variable coefficients. Appl. Math. Comput. 2004:149:15-29.
8. Biazar J, Ghazvini H. He's variational iteration method for solving hyperbolic differential equations. Int. J. Nonlinear Sci. Numer. Simul. 2007:8:311-314.
9. Sadighi A, Ganji DD. Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods. Int. J. Nonlinear Sci. Numer. Simul. 2007:8:435-443.
10. Abbasbandy S, Shivanian E. Application of variational iteration method for n th-order integro-differential equations. Z. Naturforsch. A 2009:64(a): 439-444.
11. Abbasbandy S, Shivanian E. Application of the variational iteration method for system of nonlinear Volterra's integro-differential equations. Math. Comput. Appl. 2009:14:147-158.
12. Allahviranloo T, Panahi A, Rouhparvar H. A computational method to find an approximate analytical solution for fuzzy differential equations, An. Stiint. Univ. Ovidius Constanta Ser. Mat. 2009:17:5-14.
13. Ma M, Friedman M, Kandel A. A new fuzzy arithmetic. Fuzzy Sets and Systems 1999: 108: 83-90.
14. Goetschel R, Voxman W. Elementary fuzzy calculus. Fuzzy Sets and Systems 1986:18:31-43.
15. He JH. Variational principles for some nonlinear partial differential equations with variable coefficients. Chaos, Solitons & Fractals 2004 : 19:847-851.
16. He JH. Some asymptotic methods for strongly nonlinear equations. Int. J. Mod. Phys. B 2006: 20:1141-1199.
17. Wazwaz AM. The variational iteration method for solving linear and nonlinear systems of PDEs. Comput. Math. Appl. 2007:54:895-902.
18. Buckley JJ, Qu Y. On using α -cuts to evaluate fuzzy functions. Fuzzy Sets and Systems 1990: 38:309-312.
19. Seikkala S. On the fuzzy initial value problem, Fuzzy Sets and Systems 1987:24: 319-330.

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