Profile of Minimum Drag

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Abstract: In this paper a variational integral is constructed for the estimation of the coefficient of minimum drag for axial flow over axi-symmetric bodied of revolution. The unknown equation of the profile is determined by writing and solving the corresponding Euler-Lagrange equation. This results in the equation $\frac{y}{c_1} = \frac{\left(1+\left(\frac{dx}{dy}\right)^2\right)^{\frac{3}{2}}}{-\frac{dx}{dy}}$. This reduces to a cubic equation and the real root is obtained by the method of Cardan. The equation of the curve is then obtained by integration. The integral for the Drag coefficient is computed numerically. The profile $y(x)$ is plotted graphically.

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Key Words*:* Minimum Drag, variational integral, axial, axi-symmetric.

Introduction:

In aeronautical and marine engineering Drag force on moving bodies must be kept small enough to increase the range and reduce the ejecting force (the release force). Aircrafts and submarines have stream lined shape and conical shaped fronts to ease the fluid flow over. Such bodies are bodies of revolution with their axis of symmetry along the direction of flow. The diameter of the base of the conical front is equal to the diameter of the body to ensure smooth junction. Also, the depth of this conical front is determined by other design and construction parameters.

Drag on bodies of revolution and other bodies' results from the change in momentum of the fluid stream upon attacking the body. Fluid mechanics principles assert that the resistance force $\underline{F}$ associated with mass flow rate $\dot{m}$ and caused by change in velocity of the stream $\Delta\underline{v}$ is equal to $\underline{F} = \dot{m}\,\Delta\,\underline{v}$. This force is resolved into two components; axial and named Drag D along the axis of flow and opposite to its direction $D = \dot{m}\,\Delta\,v_{axial}$ and a component called lift force L normal to the direction of flow and is given by $L = \dot{m}\,\Delta\,v_{normal}$. In bodies of revolution with axial flow, the integral of the lift force over the surface of revolution is zero due to symmetry [10, pp. 271].

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For a jet of velocity v and mass density ρ associated with projected area A, the mass flow rate $\dot{m} = \rho\,v\,A$ the change in velocity $\Delta\,v$ along the direction of v is $\Delta\,v = \frac{1}{2}c_D v$ when c_D a shape factor which is defined as the coefficient of Drag. So the drag force $D = c_D A\left(\frac{1}{2}\,\rho\,v^2\right)$, where $\frac{1}{2}\,\rho\,v^2$ is defined as stagnation pressure [10, pp. 115]. In this paper we shall obtain the equation of the profile of the head which ensures minimum drag and the corresponding drag coefficient.

Literature review

The treatment in the present work is based on two basic subjects; axi-symmetric flows and variational methods. Several articles in literature can be found on both subjects.

For the first; namely axi-symmetric flow we mention first the paper by Cumming et al. [1]in which they handled the problem of supersonic turbulent flow computations and drag optimization for axi-symmetric after-bodies. Next we mention the similarity study on mean pressure distributions of cylindrical and spherical bodies by Yeung [2]. Montes and Fernandez [3] studied the behavior of hemi-spherical dome subjected to wind loading. Also, Nelson et al. [4] determined the surface pressure for axi-symmetric bluff bodies. For the variational methods, we refer to the paper on variational methods, multi-symmetric geometry and continuum mechanics by Marsden et al. [5]. We refer also to the paper by Fernandez et al. [6] on the stress energy-momentum tensors in higher order variational calculus. Next we mention the work by Kouranbaeva and Shkoller [7] on variational approach to second order multi-symmetric field theory. At last we mention the paper by Lewis and Murray [8] on the variational principles for constrained systems. The minimum drag shape recently treated by by Dong et al.[11] deals with the problem for semi-ellipsoid exposed to shear flow but without obtaining the profile

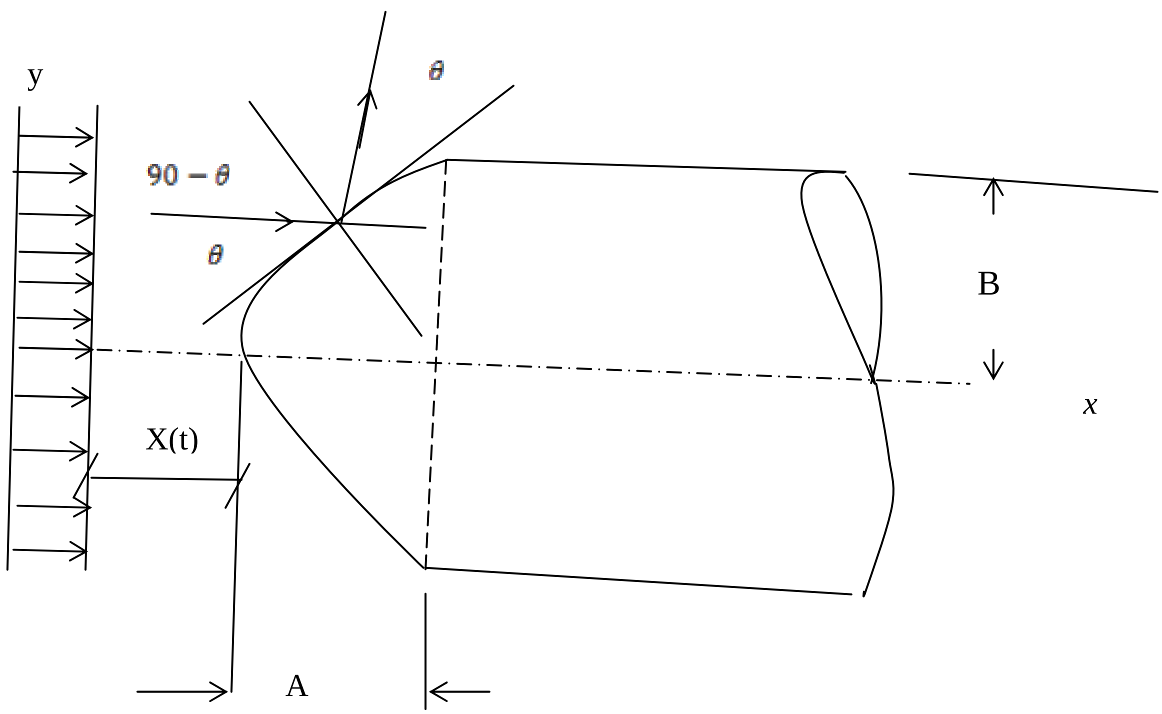

Figure 1: A schematic of the problem

Formulation of the problem

Consider the body whose axis of symmetry lies along the x axis. The body is at rest in its frame moving toward the y axis with its velocity v in the negative x direction. Uniform horizontal flow with relative positive velocity v moving towards the body. The maximum radius of revolution of the body is B and the head depth is A. The flow attacks the body and reflects on the surface of the head. We consider frictionless attack so that the angle of attack θ equals the angle of reflection. We also consider perfect attack so that the magnitude of the velocity of attack is equal to the velocity of reflection; both are equal to the relative velocity v. A schematic of the problem is shown in figure (1).

The change in the horizontal velocity due to attack

$$= v(1-\cos 2\theta) = 2\, v \sin^2\theta = \Delta\, v$$

The total mass flow rate on the body $\dot{m} = \rho\, v\, \pi\, B^2$. The element of drag force on the body $= dD = 2v \sin^2\theta\, \rho\, v\, dS$ where $dS = 2\,\pi y\, ds$;so that $D = 4\pi\, \rho\, v^2 \int_0^B \frac{dy}{ds}\, y\, dy$. Since

$\frac{dy}{ds} = \frac{1}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$, we have $\therefore\ D = 4\pi\rho v^2 \int_0^B \frac{y\,dy}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$

$$\therefore\ c_D = \frac{8}{B^2}\int_0^B \frac{y\,dy}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}} \qquad (1)$$

We are interested in finding the shape $x(y)$ or $y(x)$ which minimizes the coefficient of drag c_D

Solution

We require determining the function $y(x)$ or its inverse function $x(y)$ so that the integral $\int_0^B \frac{y\,dy}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$ is minimum. Putting

$L(y,x,x') = \frac{y}{\sqrt{\left(\frac{dx}{dy}\right)^2+1}}$,we require

$\delta\int_0^B L(y,x,x')\,dy = 0$ with the relation $x(y)$ is not yet determined. The Euler-Lagrange's equation is $\frac{\partial L}{\partial x} - \frac{d}{dy}\frac{\partial L}{\partial x'}=0$ (2)

But

$\frac{\partial L}{\partial x} = 0\ \therefore\ \frac{\partial L}{\partial x'}$ =constant$=c_1$, where $x' = \frac{dx}{dy}$

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$$\frac{c_1}{y} = -\frac{x'}{(1+x'^2)^{3/2}} \qquad (3)$$

Where y, x' are positive and c_1 is negative and has the unit of distance.

Equation (3) is rearranged to take the form $Y = \left(X'^{-\frac{2}{3}} + X'^{\frac{4}{3}}\right)^{\frac{3}{2}}$, where $Y = -\frac{y}{c_1}$ and

$X = -\frac{x}{c_1}$. Now, let $X'^{\frac{2}{3}} = v$, leading to $v^3 - Y^{\frac{2}{3}} v + 1 = 0$. From [9, pp.9], we get the real root

$$v = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{Y^2}{27}}} - \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{Y^2}{27}}} = X'^{\frac{2}{3}} \qquad (4)$$

Consequently, the ranges of Y and v will be

Which is equated to $c_D \pi B^2 \frac{1}{2} \rho v^2$ to yield

$$0 \le Y \le \frac{3\sqrt{3}}{2} = 2.598\ ,\ 1 \ge v \ge 0$$

From (4), we can write

$$X' = \left(\sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{Y^2}{27}}} - \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{Y^2}{27}}}\right)^{\frac{3}{2}}$$

Then

$$X = \int_0^Y \left(\sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{Y^2}{27}}} - \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{Y^2}{27}}}\right)^{\frac{3}{2}} dY \qquad (5)$$

The integrand is expanded in series up to degree 8 then is integrated[1] to yield

$$X(Y) = Y - \frac{3}{10}Y^{\frac{5}{3}} + \frac{1}{56}Y^{\frac{7}{3}} - \frac{7}{1296}Y^3 - \frac{83}{269568}Y^{\frac{13}{3}} - \frac{179}{1244160}Y^5 - \frac{2093}{76142592}Y^{\frac{17}{3}} + \frac{62441}{4084826112}Y^{\frac{19}{3}} - \frac{70481}{9029615616}Y^7 - \frac{35279105}{3204223598592}Y^{\frac{23}{3}} + \frac{22382693}{20897110425600}Y^{\frac{25}{3}}$$

$+O(Y^9)$ (6)

The maximum error is less than 10^{-3}, for the range $0 \le Y \le \frac{3\sqrt{3}}{2} = 2.598$ and $0 \le X \le 1.173$. This function isplotted in figure 2. The profile curve doesn't pass through (0,0), since X' is not defined at Y=0

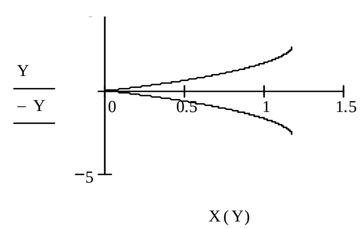

figure 2.a

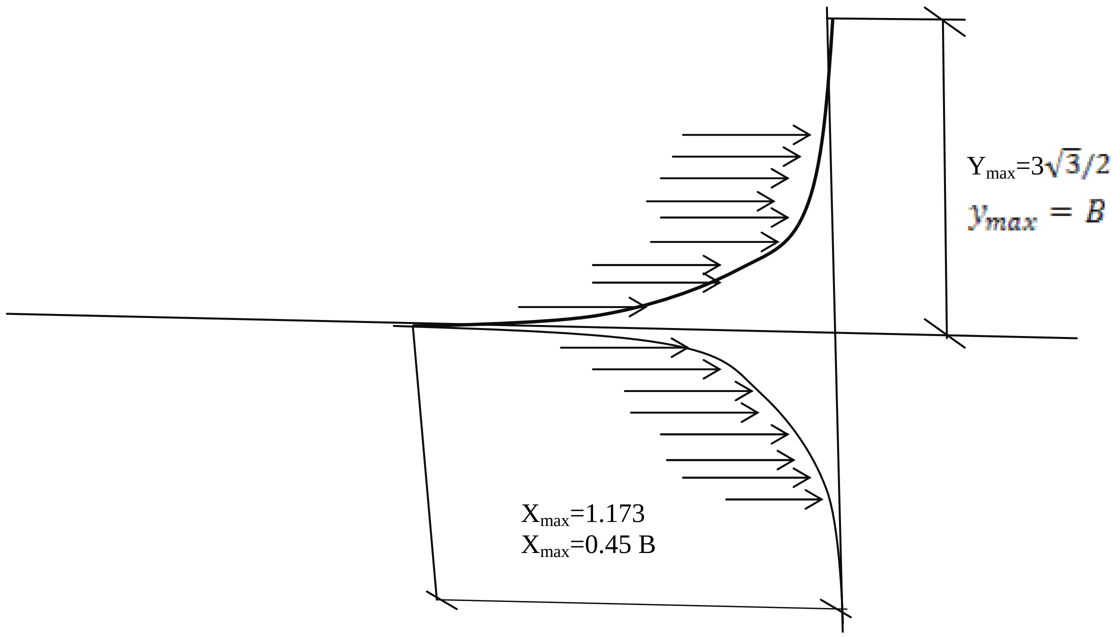

Figure 2 b: Schematic for major dimensions.

The depth of the profile

$y_{max} = -c_1 Y_{max} = B$, then $c_1 = -\frac{2B}{3\sqrt{3}}$

$X_{max} = X(Y_{max}) = 1.173$

$x_{max} = -c_1 X_{max} = 1.173\frac{2B}{3\sqrt{3}} = 0.45$ B

The drag coefficient:

Using equation (1)

$$c_D = \frac{8}{B^2}\int_0^B \frac{y\,dy}{\sqrt{x'^2+1}} = \frac{32}{27}\int_0^{3\sqrt{3}/2} \frac{Y\,dY}{\sqrt{X'^2+1}}$$

We recall the value of X'^2 from equation (4), the integrand is finite in the whole range of integration. The above integral is computed using Simpson's rule with 1000 subdivisions and gives $c_D = 3.78$

Comment

The analysis is carried out for invicid flow in the absence of any friction; this requires that the velocities are small enough to prevent turbulence. Very little is available in literature for invicid drag on axi-symmetric bodies; so, comparison with other results was found difficult. The minimum value of the coefficient of drag for axi-symmetric bodies is found to be 3.78. The value of the drag coefficient for spheres is found experimentally 4 at Reynolds number 10 [10,pp. 271].

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