Fractional Differential Equations with Fuzzy Order

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Abstract: In this paper we introduce fractional differential equations with fuzzy order. Then using Variational Iteration method we propose a method for computing approximations of solution of fractional differential equations with fuzzy order.

Key words: Fuzzy number, Fractional derivative, Variational iteration method.

1. Introduction

Fractional differential equations have been the focus of many studies. So many attention has been given to the solution of fractional differential equations and a number of literatures concerning the application of fractional differential equations in nonlinear dynamics has been grown recently [13-15]. Fractional differentials with uncertainty in parameters or initial values, have been the recent study of authors as fuzzy fractional initial value problems [3,6]. Fractional differential equations with uncertainty in the order of fractional derivative will form fractional differential equations with fuzzy order which is introduced in this paper. Fractional differential equations which arise in real-world physical problems are often too complicated to solve exactly. We propose a method for computing approximations of solution of a fractional differential equations with fuzzy order. The Variational Iteration method has been shown [1,2] to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate crisp solutions in crisp problems. Here we use the advantage of this method to find an approximate solution for the fractional differential equations with fuzzy order.

2. Preliminaries

Some basic definitions will be considered as follows.

Definition 2.1. We write $A(x)$, a number in $[0,1]$, for the membership function of $A$ evaluated at $x$. For $0 < \alpha \leq 1$ an $\alpha$-cut of $A$ written $A_{\alpha}$ is defined as $\{x | A(x) \geq \alpha \}$, also $A_{0}$ is defined as the closure of the union of all the $A_{\alpha}$, $0 < \alpha \leq 1$.

The parametric form of a fuzzy number can be defined as follows. According to the representation theorem for fuzzy numbers or intervals [9], we use $\alpha$-cut setting to define a fuzzy number or interval.

Definition 2.2 [9]. A fuzzy number (or interval) $u$ is completely determined by any pair $u = (u_1, u_2)$ of functions $u_{1,2} : [0,1] \rightarrow \mathbb{R}$, defining the end-points of the $\alpha$-cuts, satisfying the three conditions:

(i) $u_{1} : \alpha \rightarrow u_{1}(\alpha) \in \mathbb{R}$ is a bounded monotonic increasing (non-decreasing) left-continuous function $\forall \alpha \in (0,1]$ and right-continuous for $\alpha = 0$;

(ii) $u_{2} : \alpha \rightarrow u_{2}(\alpha) \in \mathbb{R}$ is a bounded monotonic decreasing (non-increasing) left-continuous function $\forall \alpha \in (0,1]$ and right continuous for $\alpha = 0$;

(iii) $u_{1}(\alpha) \leq u_{2}(\alpha) \quad \forall \alpha \in [0,1]$.

If $u_{1}(\alpha) < u_{2}(\alpha)$ we have a fuzzy interval and if $u_{1}(\alpha) = u_{2}(\alpha)$ we have a fuzzy number; for simplicity we refer to fuzzy numbers as intervals. We will then consider fuzzy numbers of normal and upper semicontinuous form also we assume that the support $[u_{1}(\alpha), u_{2}(\alpha)]$ of $u$ is compact (closed and bounded). The notation $u_{\alpha} = [u_{1}(\alpha), u_{2}(\alpha)]$, $\alpha \in [0,1]$ denotes explicitly the $\alpha$-cuts of $u$.

Definition 2.3 We say that the fuzzy number $A = (A_{1}, A_{2})$ is not less than zero, and write $A \geq 0$ iff $A_{1} \geq 0$ and $A_{2} \geq 0$.

hence for two ordered fuzzy numbers $A$, $B$ the relation $A \geq B$ holds if $A - B \geq 0$, which makes $F$ a partial ordered ring. In the situation when for two numbers $A$, $B$ the above inequality holds and, moreover, $A \neq B$ we will write $A > B$. According to the above denotations we can write shortly $A < \hat{1}$ for the case when $\hat{1} - A > 0$.

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Definition 2.4 A mapping \( F : I \to E \) is strongly measurable if for all \( \alpha \), \((0 < \alpha \leq 1)\), \( F_\alpha(t) \) is Lebesgue measurable for any \( t \in I \), where \( F_\alpha(t) = [F(t)]^\alpha \).

Definition 2.5 A mapping \( F : I \to E \) is called levelwise continuous at \( t_0 \in I \) if all its \( \alpha \)-levels \( F_\alpha(t) = [F(t)]^\alpha \) are continuous at \( t = t_0 \) with respect to the metric \( d \). And \( F \) is called integrably bounded if there exist an integrable function \( h \) such that \( |x| \leq h(t) \) for all \( x \in [F(t)]^\alpha \).

We define \( F_\alpha(t) = [F(t)]^\alpha = [F_1(t, \alpha), F_2(t, \alpha)] \) as \( \alpha \)-levels of the mapping \( F : I \to E \). A function \( f : I \to R \) is said to be a measurable selection of \( F_\alpha(t) \) if \( f(t) \) is measurable and \( F_\alpha(t, \alpha) \leq f(t) \leq F_\alpha(t, \alpha) \) for all \( t \in I \).

Definition 2.6 Let \( F : I \to E \). The integral of \( F \) over \( I \) is defined levelwise by the equation \( \int_I F(t)dt = \int_I F_\alpha(t)dt = \{ \int_I f(t)dt \mid f : I \to R \) is a measurable selection for \( F_\alpha \} \) for all \( 0 < \alpha \leq 1 \).

3. Fractional derivative of fuzzy exponent

If \( r \in \mathbb{R} \), we use the following fractional derivative and extend it to a kind of fuzzy derivative when \( r \in E \) is a fuzzy number.

Definition 3.1 Caputo fractional derivative of order \( r \) \((0 < r < 1)\) for \( u(t) : R \to R \) is defined as

\[
D_r^u = \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} u'(s)ds
\]

If \( r \in \mathbb{R} \) then \( \Gamma(r) \in \mathbb{R} \), now \( r \in E \) is represented by a pair of continuous functions, say \( r = (r_1, r_2) \), then its composition with the Euler function \( \Gamma \) leads to ordered fuzzy number

\[
\Gamma(r)(s) = \Gamma(r_1)(s, \Gamma(r_2)(s)), \quad s \in [0,1]
\]

which is just a pair of continuous function of \( s \) variable, the composition of real-valued function \( \Gamma \) with an ordered fuzzy number \( r = (r_1, r_2) \) gives the pair \( (\Gamma(r_1)(s), \Gamma(r_2)(s)) \in \mathbb{E} \) which is an ordered fuzzy number.

Definition 3.2 By a fuzzy fractional derivative of order \( r \in E \) of \( y \) we understand a function defined on \([0,\infty) = \mathbb{R}_+\) with its value in \( E \), i.e. for each \( t \in [0,\infty) \) it is an ordered fuzzy number given by the classical Caputo definition for fractional derivative of order \( r \) of the function \( x(t) \)

\[
\frac{d^r x(t)}{dt^r} = \frac{1}{\Gamma(n-r)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{n-r}} d\tau
\]

for \( n-1 < r \leq n \). If \( \alpha = r \) then

\[
\frac{d^\alpha x(t)}{dt^{\alpha}} = x^{(\alpha)}(t)
\]

here \( x^{(\alpha)}(\tau) \) denotes the \( \alpha \)-order derivative of the function \( x(\tau), \tau \in \mathbb{R}^+ \).

In preceding definition the inequality \( n-1 < r \leq n \) is understood as a relation in \( E \) between ordered fuzzy numbers: \( n-1, \alpha \) and \( n \), i.e. for each \( s \in [0,1] \)

\( n-r_2(s) \geq 0 \) and \( r_2(s) - (n-1) > 0 \) for the up-part of \( r \), and \( r_1(s) - (n-1) > 0 \) and \( n-r_1(s) \geq 0 \) for the down-part of ordered fuzzy number \( r \), where \( s \in [0,1] \). Since for \( \Gamma(r) \) we will have the representation (7) and for \( x^r = \frac{d^r x(t)}{dt^r} \), we obtain a new representation \( (x_1^r, x_2^r) \) as a pair of function of two variables \( t \) and \( s \), given by

\[
x_1^r(t, \alpha) = \frac{1}{\Gamma(n-r_2(\alpha))} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{n-1-r_2(\alpha)}} d\tau
\]

and

\[
x_2^r(t, \alpha) = \frac{1}{\Gamma(n-r_1(\alpha))} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{n-1-r_1(\alpha)}} d\tau.
\]

4. Differential equation with fuzzy order

Consider the initial value problem with uncertainty in the order of differentiation

\[
x^{(r)}(t) = f(t, x(t)), \quad x(0) = x_0, \quad x_0, r \in E \quad (1)
\]

Now we transform Eq. (1) to a system of crisp equations as

\[
\begin{align*}
x_1^r(t, \alpha) &= f_1(t, x_1(t, \alpha), x_2(t, \alpha)), \\
x_2^r(t, \alpha) &= f_2(t, x_1(t, \alpha), x_2(t, \alpha))
\end{align*}
\]

Each fractional differential equation will be solved with variational iteration method.
To perform variational iteration method, we consider the following general differential equation

\[
Lu + Nu = g(t)
\]
where \( L \) is a linear operator, \( N \) a nonlinear operator and \( g(t) \) an inhomogeneous or forcing term. According to the variational iteration method, we can construct a correctional functional as follows:

\[
u_{n+1}(t) = \nu_n(t) + \int_0^t \lambda \{ L \nu_n(s) + N \tilde{\nu}_n(s) - g(t) \} ds,
\]

where \( \lambda \) is a general Lagrange multiplier which can be identified optimally via the variational theory. The subscript \( n \) denotes the \( n \)th approximation and \( \tilde{\nu}_n \) considered as a restricted variation, i.e. \( \Delta \tilde{\nu}_n = 0 \).

In this paper the nonlinear part (\( Nu \)), in the Eq. (2) are \( x'_1(t, \alpha) \) and \( x'_2(t, \alpha) \). As such, the corresponding correctional functional are

\[
x_{1,n+1}(t, \alpha) = x_{1,n}(t, \alpha) + \int_0^t \lambda \{ x'_1(t, \alpha) \}, x_{2,n}(t, \alpha) \) \},
\]

\[
x_1(t_0, \alpha) = x_{10,0}^{\alpha}
\]

\[
x_{2,n+1}(t, \alpha) = x_{2,n}(t, \alpha) + \int_0^t \lambda \{ x'_2(t, \alpha) \}, x_{2,n}(t, \alpha) \) \},
\]

\[
x_2(t_0, \alpha) = x_{20,0}^{\alpha}
\]

Since there exists no derivative with integer order in the foregoing correctional functionals, so there exist no way to obtain the stationary conditions directly from a functional with Riemann-Liouville’s fractional integral. Failing to determine the Lagrange multiplier, in order to identify approximately the multiplier, some approximation must be made. To calculate the approximate Lagrange multiplier, we use two integer values \( k_1 = n - 1 \) and \( k_2 = n \) to find \( \mu_1 \) and \( \mu_2 \), respectively. For instance, in the first equation of system (2) we have

\[
x_{1,n+1}(t, \alpha) = x_{1,n}(t, \alpha) + \int_0^t \mu_1 \{ x_{1,n}^{(k_1)}(t, \alpha) - f_1(t, x_{1,n}(t, \alpha), x_{2,n}(t, \alpha)) \} dt,
\]

and

\[
x_{1,n+1}(t, \alpha) = x_{1,n}(t, \alpha) + \int_0^t \mu_2 \{ x_{1,n}^{(k_2)}(t, \alpha) - f_1(t, x_{1,n}(t, \alpha), x_{2,n}(t, \alpha)) \} dt.
\]

Finally, we put \( \lambda = \beta_1 \mu_1 + \beta_2 \mu_2 \), where \( \beta_1 \) and \( \beta_2 \) are weighted factors with \( \beta_1 + \beta_2 = 1 \).

In each case, the system of crisp fractional initial value problems will be solved by variational iteration method. Finally, it must be verified whether the results make \( \alpha \)-levels of a fuzzy number.

5. Conclusion

Fractional differential equations with fuzzy order have been introduced. Variational iteration method has been applied to obtain approximate fuzzy solution.

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