Fractional Differential Equations with Fuzzy Order

Azam Noorafkan Zanjani[*,1] , Abdorreza Panahi[1]

[1.] Department of Mathematics, Islamic Azad University, Saveh Branch, Saveh, Iran.
Azam57@gmail.com

Abstract: In this paper we introduce fractional differential equations with fuzzy order. Then using Variational iteration method we propose a method for computing approximations of solution of fractional differential equations with fuzzy order.
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1. Introduction

Fractional differential equations have been the focus of many studies. So many attention has been given to the solution of fractional differential equations and a number of literatures concerning the application of fractinal differential equations in nonlinear dynamics has been grown recently [13-15]. Fractional differentials with uncertainty in parameters or initial values, have been the recent study of authors as fuzzy fractional initial value problems [3,6]. Fractional differential equations with uncertainty in the order of fractional derivative will form fractional differential equations with fuzzy order which is introduced in this paper. Fractional differential equations which arise in real-word physical problems are often too complicated to solve exactly. We propose a method for computing approximations of solution of a fractinal differential equations with fuzzy order. The Variational Iteratiion method has been shown [1,2] to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate crisp solutions in crisp problems. Here we use the advantage of this method to find an approximate solution for the fractional differential equations with fuzzy order.

2. Preliminaries

Some basic definitions will be considered as follows. Let E denote the class of fuzzy sets on the real line.

Definition 2.1. We write $A(x)$, a number in $[0,1]$, for the membership function of A evaluated at x. For $0<\alpha\le 1$ an α-cut of A written A_α is defined as $\{x \mid A(x)\ge\alpha\}$, also A_0 is defined as the closure of the union of all the A_α, $0<\alpha\le 1$.

The parametric form of a fuzzy number can be defined as follows. According to the representation theorem for fuzzy numbers or intervals [9], we use α-cut setting to define a fuzzy number or interval.

Definition 2.2 [9]. A fuzzy number (or interval) u is completely determined by any pair $u=(u_1,u_2)$ of functions $u_{1,2}:[0,1]\to R$, defining the end-points of the α-cuts, satisfying the three conditions:

(i) $u_1:\alpha\to u_1(\alpha)\in R$ is a bounded monotonic increasing (non-decreasing) left-continuous function $\forall\alpha\in(0,1]$ and right-continuous for $\alpha=0$;

(ii) $u_2:\alpha\to u_2(\alpha)\in R$ is a bounded monotonic decreasing (non-increasing) left-continuous function $\forall\alpha\in(0,1]$ and right continuous for $\alpha=0$;

(iii) $u_1(\alpha)\le u_2(\alpha)\ \ \forall\alpha\in[0,1]$.

If $u_1(\alpha)<u_2(\alpha)$ we have a fuzzy interval and if $u_1(\alpha)=u_2(\alpha)$ we have a fuzzy number; for simplicity we refer to fuzzy numbers as intervals.

We will then consider fuzzy numbers of normal and upper semicontinuous form also we assume that the support $[u_1(\alpha),u_2(\alpha)]$ of u is compact (closed and bounded). The notation $u_\alpha=[u_1(\alpha),u_2(\alpha)]$, $\alpha\in[0,1]$ denotes explicitly the α-cuts of u.

Definition 2.3 We say that the fuzzy number $A=(A_1,A_2)$ is not less than zero, and write

$$A\ge 0 \text{ iff } A_1\ge 0 \text{ and } A_2\ge 0.$$

hence for two ordered fuzzy numbers A, B the relation $A\ge B$ holds if $A-B\ge 0$, which makes F a partial ordered ring. In the situation when for two numbers A, B the above inequality holds and, moreover, $A\ne B$ we will write $A>B$. According to the above denotations we can write shortly $A<\hat{1}$ for the case when $\hat{1}-A>0$.

Definition 2.4 A mapping $F: I \to E$ is strongly measurable if for all α, ($0 < \alpha \le 1$), $F_\alpha(t)$ is Lebesgue measurable for any $t \in I$, where $F_\alpha(t) = [F(t)]^\alpha$.

Definition 2.5 A mapping $F: I \to E$ is called levelwise continuous at $t_0 \in I$ if all its α-levels $F_\alpha(t) = [F(t)]^\alpha$ are continuous at $t = t_0$ with respect to the metric d. And F is called integrably bounded if there exist an integrable function h such that $|x| \le h(t)$ for all $x \in [F(t)]^0$.

We define $F_\alpha(t) = [F(t)]^\alpha = [F_1(t,\alpha), F_2(t,\alpha]$ as α-levels of the mapping $F: I \to E$. A function $f: I \to R$ is said to be a measurable selection of $F_\alpha(t)$ if $f(t)$ is measurable and

$F_1(t,\alpha) \le f(t) \le F_2(t,\alpha)$

for all $t \in I$.

Definition 2.6 Let $F: I \to E$. The integral of F over I is defined levelwise by the equation

$[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt = \{\int_I f(t)dt \mid f: I \to R$ is a measurable selection for $F_\alpha\}$,

for all $0 < \alpha \le 1$.

3. Fractional derivative of fuzzy exponent

If $r \in \Re$, we use the following fractional derivative and extend it to a kind of fuzzy derivative when $r \in E$ is a fuzzy number.

Definition 3.1 Caputo fractional derivative of order r ($0 < r < 1$) for $u(t): R \to R$ is defined as

$$D_c^{\,r} u = \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} (\frac{d}{ds} u(s)) ds$$

If $r \in \Re$ then $\Gamma(r) \in \Re$, now $r \in E$ is represented by a pair of continuous functions, say

$$r = (r_1, r_2)$$

then its composition with the Euler function Γ leads to ordered fuzzy number

$$\Gamma(r)(s) = (\Gamma(r_1)(s), \Gamma(r_2)(s)) \quad , \quad s \in [0,1]$$

which is just a pair of continuous function of s variable, the composition of real-valued function Γ with an ordered fuzzy number $r = (r_1, r_2)$ gives the pair $(\Gamma(r_1)(s), \Gamma(r_2)(s)) \in E$ which is an ordered fuzzy number.

Definition 3.2 By a fuzzy fractional derivative of order $r \in E$ of y we understand a function defined on $[0,\infty) =: \Re^+$ with its value in E, i.e. for each $t \in [0,\infty)$ it is an ordered fuzzy number given by the classical Caputo definition for fractional derivative of order r of the function $x(t)$

$$\frac{d^r x(t)}{dt^r} = \frac{1}{\Gamma(n-r)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{r+1-n}} d\tau$$

for $n-1 < r \le n$. If $\alpha = r$ then

$$\frac{d^r x(t)}{dt^r} = x^{(n)}(\tau)$$

here $x^{(n)}(\tau)$ denotes the $n-$order derivative of the function $x(\tau)$, $\tau \in \Re^+$.

In preceding definition the inequality $n-1 < r \le n$ is understood as a relation in E between ordered fuzzy numbers: $n-1$, α and n, i.e. for each $s \in [0,1]$ $n - r_2(s) \ge 0$ and $r_2(s) - (n-1) > 0$ for the up-part of r, and $r_1(s) - (n-1) > 0$ and $n - r_1(s) \ge 0$ for the down-part of ordered fuzzy number r, where $s \in [0,1]$. Since for $\Gamma(r)$ we will have the representation (7) and for $x^r =: \frac{d^r x(t)}{dt^r}$, we obtain a new representation (x_1^r, x_2^r) as a pair of function of two variables t and s, given by

$$x_2^r(t,\alpha) = \frac{1}{\Gamma(n - r_2(\alpha))} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{r_2(\alpha)+1-n}} d\tau$$

and

$$x_1^r(t,\alpha) = \frac{1}{\Gamma(n - r_1(\alpha))} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{r_1(\alpha)+1-n}} d\tau.$$

4. Differential equation with fuzzy order

Consider the initial value problem with uncertainty in the order of differentiation

$$x^{(r)}(t) = f(t, x(t)) \ , \quad x(0) = x_0 \quad , \ x_0, r \in E. \qquad (1)$$

Now we transform Eq. (1) to a system of crisp equations as

$$\begin{cases} x_1^r(t,\alpha) = f_1(t, x_1(t,\alpha), x_2(t,\alpha)), \\ x_1(t_0,\alpha) = x_1^{0,\alpha} \\ x_2^r(t,\alpha) = f_2(t, x_1(t,\alpha), x_2(t,\alpha)), \\ x_2(t_0,\alpha) = x_2^{0,\alpha} \end{cases} \qquad (2)$$

Each fractional differential equation will be solved with variational iteration method.

To perform variational itaration method, we consider the following general differential equation

$Lu + Nu = g(t)$

where L is a linear operator , N a nonlinear operator and $g(t)$ an inhomogeneous or forcing term. According to the variational iteration method, we can construct a correctional functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{Lu_n(s) + N\tilde{u}_n(s) - g(t)\}ds,$$

where λ is a general Lagrange multiplier which can be identified optimally via the variational theory. The subscript n denotes the n^{th} approximation and $\tilde{u}_n$ considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

In this paper the nonlinear part (Nu), in the Eq. (2) are $x_1^r(t,\alpha)$ and $x_2^r(t,\alpha)$. As such, the corresponding correctional functional are

$$\begin{cases} x_{1,n+1}(t,\alpha) = x_{1,n}(t,\alpha) + I^r(\lambda_1\{x_{1,n}^r(t,\alpha) \\ \qquad\qquad - f_1(t, x_{1,n}(t,\alpha), x_{2,n}(t,\alpha))\}), \\ x_1(t_0,\alpha) = x_1^{0,\alpha} \\ x_{2,n+1}(t,\alpha) = x_{2,n}(t,\alpha) + I^r(\lambda_2\{x_{2,n}^r(t,\alpha) \\ \qquad\qquad - f_2(t, x_{1,n}(t,\alpha), x_{2,n}(t,\alpha))\}), \\ x_2(t_0,\alpha) = x_2^{0,\alpha}. \end{cases}$$

Since there exists no derivative with integer order in the foregoing correctional functionals, so there exist no way to obtain the stationary conditions directly from a functional with Riemann-Liouville's fractional integrate. Failing to determine the Lagrange multiplier, in order to identify approximately the multiplier, some approximation must be made. To calculate the approximate Lagrange multiplier, we use two integer values $k_1 = n-1$ and $k_2 = n$ to find μ_1 and μ_2 respectively. For instance, in the first equation of system (2) we have

$$x_{1,n+1}(t,\alpha) = x_{1,n}(t,\alpha) + \int_0^t(\mu_1\{x_{1,n}^{(k_1)}(t,\alpha) - f_1(t, x_{1,n}(t,\alpha), x_{2,n}(t,\alpha))\})dt$$

and

$$x_{1,n+1}(t,\alpha) = x_{1,n}(t,\alpha) + \int_0^t(\mu_2\{x_{1,n}^{(k_2)}(t,\alpha) - f_1(t, x_{1,n}(t,\alpha), x_{2,n}(t,\alpha))\})dt.$$

Finally, we put $\lambda_1 = \beta_1\mu_1 + \beta_2\mu_2$, where β_1 and β_2 are weighted factors with $\beta_1 + \beta_2 = 1$.

In each case, the system of crisp fractional initial value problems will be solved by variational iteration method. Finally, it must be verified whether the results make α -levels of a fuzzy number.

5. Conclusion

Fractional differential equations with fuzzy order have been introduced. Variational iteration method has been applied to obtain approximate fuzzy solution.

***Corresponding Author:**
Azam Noorafkan Zanjani
Department of Mathematics
Islamic Azad University
Saveh Branch, Saveh, Iran
E-mail: Azam57@gmail.com

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