Efficient radial basis functions collocation methods for numerical solution of the parabolic PDE's

Majid Amirfakhrian

Department of Mathematics, Islamic Azad University, Central Tehran Branch,
Payambar Complex, Shahrak Gharb,Tehran,Iran.
majiamir@yahoo.com, amirfakhrian@iauctb.ac.ir

Abstract: In this paper, we apply the collocation methods of meshfree RBF over differential equation containing partial derivation of one dimension time dependent with a compound boundary nonlocal condition. in this work, we compare efficient collocation methods in order to obtain approximate solution of nonlocal parabolic differential equations.

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1. Introduction

In the recent decades the meshless methods became more considerable for their good flexibility in the finite dimension spaces; therefore the collocation of RBF methods are always successful [6,15] when they obtained suitable results over some types of differential equations containing time dependent partial derivations. If we apply special precision in selection of radial basis function, PDE equation solution can be approximated well, too.

In this paper, we focus on the solution of one dimensional and nonclassical parabolic time dependent differential equations which they have a mixture of integral term and ξ th-derivation of an unknown function u in nonlocal boundary conditions where $\xi \in \{0,1\}$.

Here, we try to solve the above equation by different RBF collocation methods.

Consider the following non-classic boundary value parabolic equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + q(x,t),$$

for $0 < x < l, \quad 0 < t \le T, \; \alpha > 0.$

with initial condition

$$u(x,0) = f(x), \; 0 \le x \le l. \tag{1}$$

and boundary nonlocal conditions

$$\frac{\partial^{\xi} u}{\partial x^{\xi}}(x_1,t) = g(t) \tag{2}$$

and

$$\int_0^{b(t)} u(x,t)dx + \beta \frac{\partial^{\zeta} u}{\partial x^{\zeta}}(x_2,t) = h(t),$$

where $\zeta, \xi \in \{0,1\}$, $x_1, x_2 \in [0,l]$, $\beta \in \mathrm{R}$ and the functions f, g, h, are all known and $0 < t \le T, \; 0 < b(t) \le l$.

In the cases that $b(t)$ is a constant function, $q(x,t) = 0$, and $x_1 = l$, $\beta = \xi = 0$ an implicit finite difference method is offered for its numerical solution by Cannon and Vander Hoek [3,4].

This equation has been solved by Dehghan and Tatari [17] with the boundary condition $\beta = \xi = 0$, $x_1 \in \{0,1\}$ and constant function $b(t)$ by GA-RBF, and then improved by them [7,18]. Chen et. all also used radial basis function for PDE's in [5]. Some papers have been written to solve partial differential equations using the collocation method with radial basis functions [8,9,10,13,14,16]. Fornberg and Piret used radial basis function for solving a convective partial differential equations on a sphere [12].

We introduce some basic concepts of radial basis functions in Section2, then we try to impose them on the mentioned differential equations. In Section 3, we give some numerical examples and numerical results of these examples. Finally, we present the Conclusion in Section 4.

2. Radial Basis Functions Knowledge

For a fixed point $\mathbf{x}_j \in \mathrm{R}^d$, a radial basis function is defined as:

$$\varphi_j(\mathbf{x}) = \varphi(\|\mathbf{x} - \mathbf{x}_j\|) \tag{3}$$

which is a function only dependent on the distance between $x \in R^d$ and the point x_j. This function is radially symmetric near the center x_j. Throughout of this work we use GA, MQ, IMQ and IQ-RBF with the following forms, respectively

$$\phi(r) = \exp(-cr^2) \quad (4)$$

$$\phi(r) = \sqrt{r^2 + c^2} \quad (5)$$

$$\phi(r) = (\sqrt{r^2 + c^2})^{-1} \quad (6)$$

$$\phi(r) = (r^2 + c^2)^{-1} \quad (7)$$

where c is a shape parameter which should be considered suitably also the Euclidean distance is considered for the RBF.These types of functions have a global support and they are suitable for interpolation of scattered data [2,11] .

3. Solving PDE's by Radial Basis Functions

Consider the following problem:

$$Lu(x,t) = q(x,t), \quad (x,t) \in \Omega \times (0,T] \quad (8)$$

which Ω is a spacial domain and L is a second order linear parabolic operator. Also consider the initial and non-local boundary operators, respectively as follows:

$$\ell u(x,t) = f(x), \quad (x,t) \in \Omega \times \{0\}$$

and for $(x,t) \in \partial\Omega \times (0,T)$ we have two operatores

$$S_\nu u(x,t) = \begin{cases} g(t), & \nu = 1, \\ h(t), & \nu = 2, \end{cases} \quad (9)$$

By considering:

$$\mathbf{x} = (x,t) \quad (10)$$

we try to find an approximating function $p(\mathbf{x})$ over R^2 for the solution of the problem in the following form:

$$p(\mathbf{x}) = \sum_{j=1}^{m} \lambda_j \varphi(\|\mathbf{x} - \mathbf{x}_j\|) \quad (11)$$

where λ_j 's are real coefficients.

Suppose that the following sets contain a collocation of scattered nodes in every levels of interpolation for $T > T_1$

$$\Xi_1 = \{(x_i, t_i) \in \overline{\Omega} \times [0, T_1], \ i = 1, \dots, m\},$$

and for $k = 2,3,\dots$ and for all $(x_i, t_i) \in \Xi_1$, $i = 1, \dots, m$

$$\Xi_k = \{(x_i, t_i + (k-1)T_1)\}$$

and the problem has a solution in $\overline{\Omega} \times [(k-1)T_1, kT_1]$.

Considering:

$$h_{ij} = \left[\frac{\partial \varphi}{\partial t} - \alpha \frac{\partial^2 \varphi}{\partial x^2}\right](\|\mathbf{x}_i - \mathbf{x}_j\|)$$

and

$$q_i = q(\mathbf{x}_i), \ i = 1, \dots, N_1$$

we have a linear system of equations

$$H\Lambda = q$$

where $H = [h_{ij}]$, $\Lambda = [\lambda_j]$

Also applying initial operator lead us to an others linear system of equations:

$$Z\Lambda = f$$

where $Z = [z_{ij}]$, such that for $j = 1, \dots, m$ and $i = N_1 + 1, \dots, N_2$

$$z_{ij} = \ell\varphi(\|\mathbf{x}_i - \mathbf{x}_j\|),$$

Now for $j = 1, \dots, m$ and $i = N_2 + 1, \dots, N_3$ we define

$$v_{ij} = S_1\varphi(\|\mathbf{x}_i - \mathbf{x}_j\|),$$

and for $j = 1, \dots, m$ and $i = N_3 + 1, \dots, m$

$$w_{ij} = S_2\varphi(\|\mathbf{x}_i - \mathbf{x}_j\|).$$

By considering:

$$S_\nu u = g, \ S_\nu u = h$$

we have

$$V\Lambda = g,$$

and

$$W\Lambda = h.$$

By considering

$$A = [H, Z, V, W]^T,$$

$$B = [q, f, g, h]^T,$$

and

$$\Lambda = [\lambda_j]$$

We have the following linear system of equations with a $m \times m$ coefficient matrix

$$A\Lambda = B$$

A is an ill-conditional full and non-symmetric matrix which its condition number may be large. But it is strictly positive definite.

This system, besides, has always a unique solution that we can convert the matrix to the same small matrixes accompany with better and smaller condition number and represent an acceptable solution by preconditioning coefficient matrix with the help of numerical methods. Since, finding solution for large times needs a lot of operators and memory, so, if T_1 is chosen such that a small number of the collocation points provides an accurate approximation in $\overline{\Omega}\times[0,T_1]$,[4] we can solve this linear system easily by factorization methods in every level of small times accompany with discritization of space and time variables [17], because in every level of time resulted matrix have a small dimension and then computing will be done easily.

Chosen points, will produce the same matrix in every level of time that we just need decomposition of a matrix [1], Since a set of scattered nodes will use instead of mesh of whole points domain.

This advantage of mentioned collocation method will be result in computing solution in different level of time without instability.

3. Numerical Examples

In this section two examples are given which have been solved by considering IMQ, MQ, GA and IQ radial basis functions. In all examples we used the method for the first layer of equations.

Example 11

In this example, we considered the problem by using

$$b(t) = \beta = x_1 = x_2 = \xi = \zeta = 1,$$

$$\alpha = \frac{8}{\pi^2}, \quad \Omega = [0,1].$$

and

$$f(x) = \sin(\frac{\pi}{2}x), \quad 0 < x < 1,$$

$$g(t) = 0, \quad 0 < t < 1,$$

$$h(t) = \frac{2}{\pi}\exp(-2t), \quad 0 < t \le 1,$$

$$q(x,t) = 0, \quad 0 < t \le 1, \quad 0 < x < 1.$$

and $0 \le t \le 0.002$, $0 \le x \le 1$; in relations (1)-(4). By considering $m = 33$, $0 \le t \le 0.002$, $\delta = 20$, $c = 2$, $\Delta x = 0.1$ and $\Delta t = 0.001$ we have the following error functions

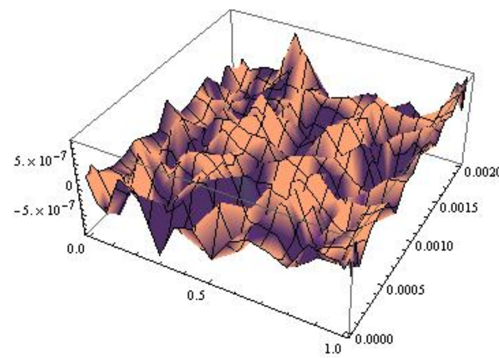

Figure 1: IMQ-RBF method

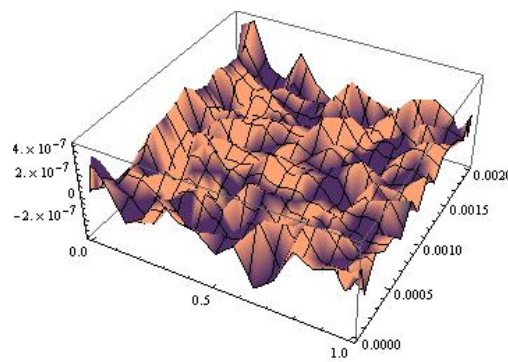

Figure 2: GA-RBF method

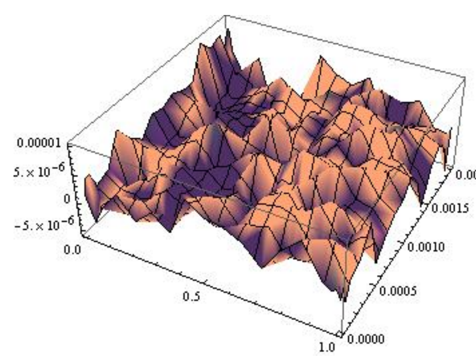

Figure 3: IQ-RBF method

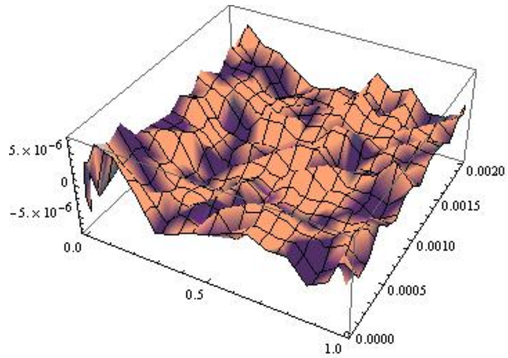

Figure 4: MQ-RBF method

The exact solution of this problem is

$$u(x,t) = \sin(\frac{\pi}{2}x)\exp(-2t) \qquad (12)$$

Example 22

In this example, we considered the problem by using

$$b(t),\ \beta,\ x_2,\ \zeta = 1,\ x_1,\ \xi = 0,$$
$$\alpha = 2, \quad \Omega = [0,1].$$

and

$$f(x) = \exp(x), \quad 0 < x < 1,$$

$$g(t) = \exp(2t), \quad 0 < t < 1,$$

$$h(t) = -\exp(2t), \quad 0 < t \le 1,$$

$$q(x,t) = 0, \quad 0 < t \le 1, \quad 0 < x < 1.$$

and $0 \le t \le 0.002$, $0 \le x \le 1$; in relations (1)-(4). By considering $m = 33$, $0 \le t \le 0.002$, $\delta = 20$, $c = 3$, $\Delta x = 0.1$ and $\Delta t = 0.001$ we have the following error functions

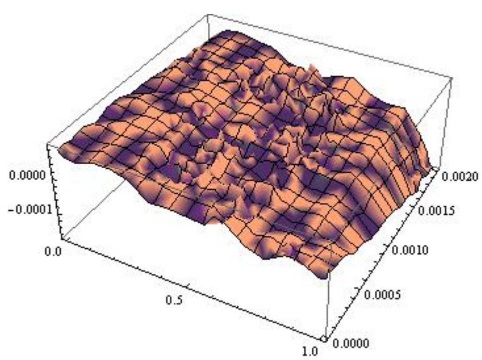

Figure 5: GA-RBF method

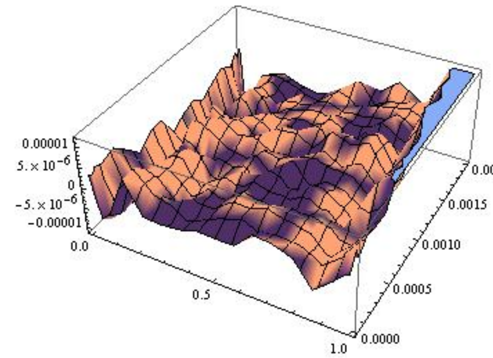

Figure 6: IMQ-RBF method

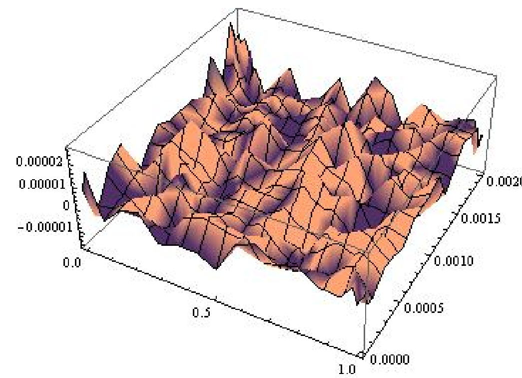

Figure 7: IQ-RBF method

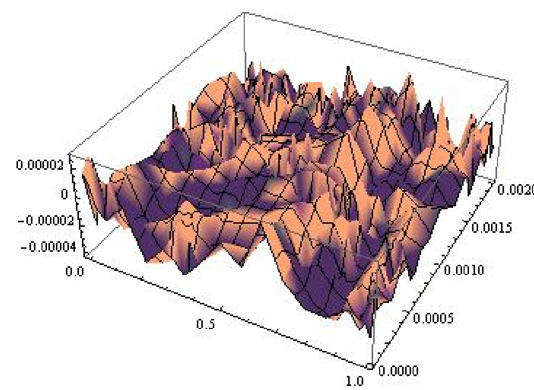

Figure 8: MQ-RBF method

The exact solution of this problem is

$$u(x,t) = \exp(x + 2t) \qquad (13)$$

3. Conclusion

In such problems, the traditional mesh methods will never offer because we can do discretization of variables by collocation methods so that the cost of computations and discretization of variables will reduce strongly. Thus, meanwhile approximating accurate and correct solution, we have at least error and we are able to change the time and space interval divisions in every level. The used radial basis function is a collocation method in this type of problems and if we take care in selection of radial basis functions approximants and their shape parameter we can obtain more accurate solution with less error.

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