On the Invertibility Preserving Linear Maps

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Abstract: In this paper we show that the essentiality of the socle of an ideal B of the algebra A implies that any invertibility preserving linear map $\Phi : A \to A$ is a Jordan homomorphism. Specially if A is a preliminary algebra then any such Φ is an algebric homomorphism.
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1. Introduction

Linear invertibility preserving maps of algebra were noteworthy from years ago, for example, the famous theorem of Glason-Kahane-Zelasco which asserts that any invertibility preserving linear isomorphism in to the scalar field is product preserving [1,14]. This problem has been discussed in different algebras previously [3,4,7,8,10,11]. Generally, A and B are two Banach algebras and we consider $\Phi : A \to B$ linier map, we know that if be a homomorphism algebra, it necessary would be invertibility preserving.

Kaplansky presented the issue as following question:

If the preserving would be invertible, is it necessarily the Jordan homomorphism? So, the main question of this paper is as Kaplansky question and we will answer it in the essential ideal of semi-simple Banach algebras.

Definition 1: the inverse of an invertible element $a \in A$ is denoted by a^{-1} and the set of all invertible elements of unitary algebra A is denoted by $Inv(A)$.

Definition 2: Let $\Phi : A \to B$, linear map between functional algebras, Φ is invertibility preserving if

$$a \in Inv(A) \Rightarrow \Phi(a) \in Inv(B) \qquad (a \in A) \quad .$$

Theorem 1: Let $\Phi : A \to B$ be a linear mapping with B commutative and semi-simple. Suppose $\varphi(Inv(A)) \subseteq Inv(B)$ and unitary maps. Then Φ is continuous and multiplicative, i.e.

$$\Phi(xy) = \Phi(x)\Phi(y) \text{ for all } x, y \in A.$$

Proof.

Let L be a multiplicative linear functional on B different from the zero functional. Then L and $L \circ \Phi$ are continuous. We show that Φ is multiplicative. It suffices to consider only the case where $B = \mathbb{C}$ since the multiplicative linear functional on B separates its points.

Thus, given $x \in A$, define $f(\lambda) = \Phi(\exp(\lambda x))$, then $f : \mathbb{C} \to \mathbb{C}$ is an entire function having no zeros since every value of the exponential function on a Banach algebra is invertible. Hence there exists an entire function with $f(\lambda) = \exp(g(\lambda))$ for all $\lambda \in \mathbb{C}$.

Moreover, $g(0) = 0$ and $\operatorname{Re}(g(\lambda)) \le |\lambda| \|x\|$ for all $\lambda \in \mathbb{C}$ and it follows from a Schwarz Lemma that $g(\lambda) = \alpha\lambda$ for some complex constant α. Thus,

$$\Phi(e + \lambda x + \frac{\lambda^2 x^2}{2!} + \cdots) = 1 + \alpha\lambda + \frac{\alpha^2\lambda^2}{2!} + \cdots \quad (\lambda \in \mathbb{C})$$

Comparing coefficients, we se that $\Phi(x) = \alpha$ and $\Phi(x^2) = \alpha^2$ so $\Phi(x^2) = \Phi^2(x)$. Define $[x, y] = xy - yx$ and $x \circ y = xy + yx$. Since Φ is a Jordan homomorphism, for any $x, y \in A$:

$$\Phi(x \circ y) = \Phi(x) \circ \Phi(y)$$

$$\Phi([x, y])^2 = \Phi([x, y]^2) = [\Phi(x), \Phi(y)]^2 = 0.$$

Because B is commutative, hence $\Phi([x, y]) = 0$ so

$$2\Phi(xy) = \Phi([x, y] + x \circ y) = 2\Phi(x)\Phi(y).$$

In the above theorem, complex field and commutative is the main provision.

Example 1: Let A real Banach algebra of all continuous real-valued functions f is on [0,1], we define

$$\Phi : A \to R$$

$$\Phi(f) = \int f(t)dt$$

Clearly A_{inv} is all function of non-zero in A and $\Phi(I) = I$, so Φ is confirmed in the above theorem, whereas Φ is not isomorphism.

Example 2: Let M_n be algebra of $n \times n$ matrices on the complex numbers, transpose function, maps will be linear and invertible, while multiplicative

$$\Phi : M_n \to M_n$$

$$\Phi(A) = A^t$$

because $\Phi(Z\,A) = (Z\,A)^t = A^t Z^t = \Phi(A)\Phi(Z)$.

Remark: Example 2 shows that an onto condition is needed.

$$\Phi : M_2 \to M_4$$

$$\Phi(Z) = \begin{bmatrix} Z & Z - Z^t \\ 0 & Z \end{bmatrix}$$

It is observed that

$$\Phi^2(Z) - \Phi(Z^2) = \begin{bmatrix} 0 & (Z - Z^t)^2 \\ 0 & 0 \end{bmatrix} \neq 0$$

In the next example, it is observed that if bijective map be invertibility preserving, semi-simple provision of algebras can not be removed.

Example 3: If Φ is bijective map and A, B Banach algebras of semi-simple, then Φ is not a Jordan isomorphism

$$\Phi : M_2 \to M_4$$

$$\Phi(\begin{bmatrix} W & X \\ 0 & Y \end{bmatrix}) = \begin{bmatrix} W & X \\ 0 & Y^t \end{bmatrix} \qquad (W, X, Y \in M_2)$$

In this case, Φ is unitary linear map, while

$$\Phi^2(Z) - \Phi(Z^2) = \begin{bmatrix} 0 & X(Y - Y^t) \\ 0 & 0 \end{bmatrix} \neq 0$$

Corollary: In 1995, Marcus-Purves showed that maps of invertibility preserving on M_n matrices are isomorphism.

2. Jordan isomorphism

In 1986, Jafarian-Souroor proved this subject on space of linear functions of $B(X)$ [7]. In 1970, Kaplansky in order to explain Theorem 1, removed commutative assumption of B. This case caused that in 1996 Kadison regarding Jafarian-Sourour's theorem being $B(X)$ semi-simple Banach algebra [7,15], tried to fined answer of Kaplansky theorem through expressing following assumption on c*- algebras in special manner.

Assumption: Suppose that A, B are two c*- algebras with the same element of c and $\Phi : A \to B$ is bijective unitary map, then is $\Phi : A \to B$ isomorphism Jordan?

Solution: While B is commutative, the first theorem proves the accuracy of above assumption.

Moreover, if B is finite of $B = L(H)$ (H is Hilbert space) or c*- algebra, compact operator on H as be the same self-addition, the above assumption will be correct.

Also, Aupetit asserts assumption in ideal that A, B be Von algebras.

Lemma 2.1: Let A be a semi-simple Banach algebra and $a \in A$, then

(i) $a \in soc(A)$ if and only if $|\sigma(xa)| < \infty \quad (x \in A)$

(ii) $a \in soc(A)$ if and only if there exists $n \in N$ such that

$$\bigcap_{t \in F} \sigma(x + ta) \subseteq \sigma(x) \qquad (x \in A)$$

for which F is the set of n-tuples of $\mathbb{C} \setminus \{0\}$.

Lemma 2.2: Let Φ be an automorphism on a semi-simple Banach algebra A, then

(i) $\Phi(soc(A)) = soc(A)$

(ii) $\Phi^{-1}(\Phi(a^2)) - \Phi^2(A).soc(A) = 0$

Proof. (i) Let $a \in A$, there exists $b \in A$ such that $\Phi(b) = a$, since Φ is spectrum preserving, then $\Phi(y) = x$ implies that

$$\sigma(\Phi(y) + ta) = \sigma(y + tb) \qquad (t \in \mathbb{C})$$

so

$$\bigcap_{t \in F} \sigma(x + t\Phi(a)) = \bigcap_{t \in F} \sigma(\Phi(y + tb)) = \sigma(x)$$

therefore, $\sigma(a) \in soc(A)$ due to part 2.3. So $\Phi(soc(A)) \subseteq soc(A)$. Now we show that $\Phi(soc(A)) \supseteq soc(A)$. If $x \in A$ then there exists $y \in A$ such that $\Phi(y) = x$ and

$$\bigcap_{t\in F} \sigma(x+tb) = \bigcap_{t\in F} \sigma(\Phi(x)+t\Phi(b)) \subseteq \sigma(y) = \sigma(x)$$

which implies that $b \in soc(A)$ due to part (ii) of Lemma 2.1.

(ii) Let $\Phi(a) = 0$ then

$$\sigma(a+x) = \sigma(\Phi(a+x)) = \sigma(x) \qquad (x \in A)$$

So $a \in Rad\, A = \{0\}$ due to Zemank theorem. Now the rest of proof is hold by Lemma 2.1.

Recall that every minimal left ideal of A is of the form Ae where e is a minimal idempotent. The sum of all minimal left ideal of A is called the socle of A and it coincides with the sum of all minimal right ideal of A. An ideal I of A is said to be essential it has a nonzero intersection with every nonzero ideal of A. If A is a semi-simple algebra, then I is essential if and only if $a.I = 0$ implies $a = 0$, where $a \in A$.

Example 4: Let H be Hilbert space and $K(H)$ be a compact operator in $B(H)$, then $K(H)$ is a essential ideal of $B(H)$.

Lemma 2.3: Let B be an ideal of A, and $soc(B)$ is an essential ideal, then $soc(A)$ is an essential ideal.

Proof. For every $a \in A$, if $a.B = 0$ then $a.soc(B) = 0$ which implies $a = 0$. Let $b \in B$ and bB be a minimal right ideal of B. Since $bB \neq 0$, there exists $b_1 \in B$ such that $bb_1 \neq 0$, we have $bb_1B \subseteq bb_1A \subseteq bB$. Because $bb_1B \neq 0$ and bB is a minimal right ideal in B, so $bb_1B = bb_1A = bB$.

If $a \in A$ and $bb_1a \neq 0$, we have $bb_1aB \subseteq bb_1aA \subseteq bb_1A \subseteq bB$ as well, because $bb_1aB \neq 0$ and bB is a minimal right ideal in B, then $bb_1aA = bb_1A = bA$. So bb_1A is a minimal right ideal in A, which implies $soc(B) \subseteq soc(A)$, if $a.soc(B) \subseteq a.soc(A) = 0$.

We know that a must be equal to zero, so $soc(A)$ is an essential ideal.

Theorem 2.3: Let B be an ideal of the semi-simple Banach algebra A and $\Phi : A \to A$ be a unitary linear isomorphism. If $soc(B)$ is an essential ideal then Φ is a Jordan isomorphism.

Proof. Let $soc(B)$ is essential, so $soc(A)$ is also an essential ideal, if A is unitary then Φ is a Jordan isomorphism by Lemma 2.2. If A is not unitary then $\tilde{A} = A \oplus \mathbb{C}$ is unitary semi-simple Banach algebra. Let $\tilde{\Phi}(a,\lambda) = (\Phi(a),\lambda)$ for $(a,\lambda) \in \tilde{A}$. Then Φ is a well defined unitary linear isomorphism. If $soc(A) = k$ and $(a,\lambda)soc(\tilde{A}) = 0$ then $soc(\tilde{A})$ is an essential ideal, since $(soc(A),0) \subseteq soc(A,0) \subseteq soc(\tilde{A})$ then $(a,\lambda)(k,0) = 0$ i.e. $ak = -\lambda k$ therefore $\lambda = 0$, because $\lambda \neq 0$ implies that $-\frac{a}{\lambda}k = k$ moreover $(-\frac{a}{\lambda}b - b)k = 0$ if and only if $-\frac{a}{\lambda}b = b$ for all $b \in A$, so $-\frac{a}{\lambda}$ is a left unit of A. Let d be left unit of A then

$$(-\frac{a}{\lambda} - d)A = 0 \Rightarrow (-\frac{a}{\lambda} - d)k = 0 \Rightarrow -\frac{a}{\lambda} = d$$

so $-\frac{a}{\lambda}$ is left unit and A is unitary which contradicts Sour hypothesis, therefore $ak = 0$. Since K is an essential ideal then $a = 0$ and Lemma 2.2 implies that

$$\tilde{\Phi}^{-1}(\tilde{\Phi}(a,\lambda)^2) - \tilde{\Phi}^2(a,\lambda)).soc(A) = 0$$

Φ is bijective

$$\tilde{\Phi}(a,\lambda)^2 = (\Phi(a^2) + 2\lambda\Phi(a), \lambda^2) \Rightarrow \Phi^2(a) = \Phi(a^2).$$

Corollary: If $\Phi : A \to A$ is a unitary linear isomorphism on the semi-simple Banach algebra A, and A has minimal ideal then $\Phi|_{soc(A)}$ is a Jordan homomorphism.

3. Conclusion

In primitive algebras every nonzero ideal is essential, and from a well-known theorem of Herstein on Jordan homomorphisms onto prime rings it follows easily that Jordan isomorphism or an anti-isomorphism.

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