Modeling Underground Water Reservoir

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Abstract: In this paper a simple model for underground water is constructed. The supply to the reservoir is from one side with variable water level from rain and flood, in the meantime water escapes outside from the other side. The soil forming the reservoir is porous and water movement inside is according to Darcy's flow. The bottom of the reservoir is impermeable to water, whereas, the top of the rectangular reservoir is exposed to a steady pressure depending on the atmospheric pressure. The differential equation of the flow in the model is solved by the method of Green's function. Inlet and exit velocity distribution is obtained and is integrated to give the capacity as a function of time.
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1. Introduction

Water is necessary for human consumption agricultural and industrial purposes. It can be found in underground reservoirs in layers mixed with soil. Typically, underground is formed of solid rock or porous soil. Walls are dug to extract, the capacity of the reservoir is essential to determine whether or not digging is economic. This model is suggested to describe the mathematical analysis employed. Empirical evidence show that flow in the porous soil is according to Darcy's law; which will be stated later. Extensive literature can be found for the flow in porous media, of this work we mention the paper by [1] on fluid flow through porous metals and the work by [2] who studied fluid flow through packed columns. More recently,[3] discussed inertial effects on fluid flow through disordered porous media. Also, [4] handled the problem of permeability of unidirectional fibrous media; this problem is also discussed by [5]. The nonlinear correction to Darcy' s law at low Reynolds numbers are made by [6]. Non-Darcy flow in porous media is studied by [7]. The relative permeability coefficients in two phase flow in porous media are studied by [8-9], the multiphase flow in porous media with phase change is discussed by [10] and [11-16] handled several aspects of multiphase flow in porous media.[17] handled multiphase flow and transport processes in subsurface. Also, [18] discussed thermodynamically constrained averaging theory for flow in porous media. For a review of underground water mechanics, we refer the reader to the text by [19] and the historical perspective by [20]. Also, classic ground water simulations are given by [21].

2. The Basic Theory

The basic theory in the flow of porous media is due to Darcy's law which is suitable for low velocity and viscosity. The theory is linear and nonlinear corrections are required of which we mention the empirical formula due to Forchheimer. Also Koch and Ladd obtained similar results with perturbation methods. The work presented in this paper is exclusively based on Darcy's flow which enables exact mathematical formulation.

Two basic assumptions are involved in Darcy's flow. The first is the existence of a linear relationship between the flow velocity and the hydrostatic head gradient, the constant of proportionality is the permeability of soil. If $\underline{u}$ is the velocity vector of the fluid, h is the hydrostatic head and $\frac{p}{\rho}$ is the equivalent head due to pressure p where ρ is the specific weight of the fluid

$$\underline{u} = k \, \underline{\nabla} \left(h + \frac{p}{\rho} \right) \qquad (1)$$

k is the permeability of soil.

The second assumption is due to the compressibility of the porous soil such that the continuity equation for incompressible water flowing in such soil is given by

$$\underline{\nabla} \cdot \underline{u} = \upsilon \frac{\partial}{\partial t} \left(h + \frac{p}{\rho} \right) \qquad (2)$$

υ is the packing constant of the soil. Denoting $\varphi = h + \frac{p}{\rho}$, eqs (1) and (2) give the diffusion equation

$$\frac{\partial \varphi}{\partial t} = \frac{k}{\upsilon} \nabla^2 \varphi \qquad (3)$$

which is the governing equation of the problem.

3. The Suggested Model

To render the mathematical analysis tractable, a simple model is suggested in Cartesian space. The water flow is in the $x-y$ plane and the analysis will be made for one depth unit normal to the flow plane. Of course, the flow is time dependent in the model considered. The inlet to the model reservoir is at $x=0$ along the y axis and the outlet is at $x=L$ also along the y axis. The bottom of the reservoir is composed of horizontal solid rock along the x axis at $y=0$ and is impermeable to water i. e. of zero vertical velocity component $\frac{\partial \varphi}{\partial y}\Big|_{y=0} = 0$.

The top of the reservoir is at $y=B$ and is also horizontal and is exposed to a fraction of the atmosphere pressure and is uniform along the top. A low hydrostatic head of a magnitude H_0 is uniformly distributed along the bottom. While a variable time dependent head αt exists on the entrance at $x=0$ along the y axis. This variable head is due to rain or flood. The fig. 1 shows a schematic of the model.

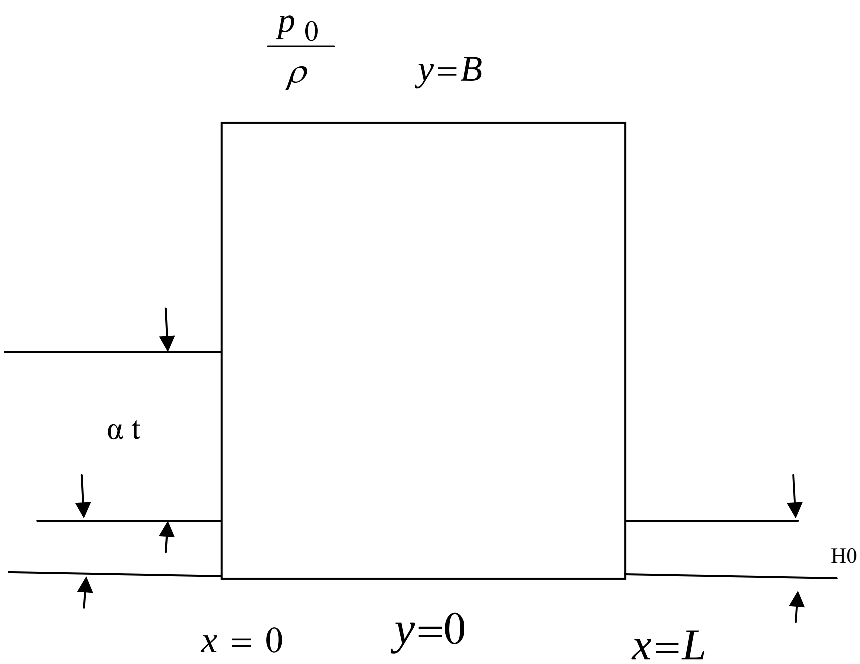

Fig. 1. Schematic of the reservoir

4. Analytical Solution

To obtain the storage capacity of the reservoir; the velocity must be obtained along the entrance and the exit and the capacity C at any instant of time T less than $\frac{B-H_0}{\alpha}$ is given by

$$C(T) = k \int_0^T \int_0^B \left[\frac{\partial \varphi}{\partial x}\Big|_{x=0} - \frac{\partial \varphi}{\partial x}\Big|_{x=L} \right] dy\,dt \qquad (4')$$

The required x derivatives of φ can only be found from the distribution of φ in the whole domain of the solution which is obtained from solving equation (3) subject to the prescribed boundary conditions as given below:

$$\varphi(0,y,t)=H_0+\frac{P_0}{\rho}+\alpha t \quad (5)$$

$$\varphi(x,B,t)=\frac{P_0}{\rho} \quad (7)$$

and

$$\frac{\partial\varphi}{\partial y}\Big|_{y=0}=0 \quad (8)$$

$$\varphi(L,y,t)=H_0+\frac{P_0}{\rho} \quad (6)$$

The integration of this time dependent boundary value problem can be obtained analytically by the method of Green's function at the typical point (x_0,y_0,t_0) [22] by the integrals:

$$\varphi(x_0,y_0,t_0)=\left(H_0+\frac{P_0}{\rho}\right)\int_0^B\int_0^L G\big|_{t=0}dxdy-\frac{k}{\nu}\int_0^T\int_0^B\left(H_0+\frac{P_0}{\rho}+\alpha t\right)\frac{\partial G}{\partial x}\Big|_{x=0}dydt$$
$$-\frac{k}{\nu}\int_0^T\int_0^B\left(H_0+\frac{P_0}{\rho}\right)\frac{\partial G}{\partial x}\Big|_{x=L}dydt-\frac{k}{\nu}\frac{P_0}{\rho}\int_0^T\int_0^L\frac{\partial G}{\partial y}\Big|_{y=B}dxdt \quad (9)$$

Here G is the Green's function of the problem and is given by solution of

$$\frac{\partial G}{\partial t}+\frac{k}{\upsilon}\nabla^2G=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \quad (10)$$

Subject to

$$G=0 \text{ on } x=0,\ x=L,\ y=B \text{ and } \frac{\partial G}{\partial y}\Big|_{y=0}=0 \quad (11)$$

The solution of eqs. (6) and (7) is obtained as

$$G(x,y,t,x_0,y_0,t_0)=\frac{4}{LB}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sin\frac{n\pi x_0}{L}\cos\frac{\pi}{2}(2m+1)\frac{y_0}{B}\cdot$$
$$\times e^{-\frac{k}{\nu}\left(\frac{n^2\pi^2}{L^2}+\pi^2\frac{(2m+1)^2}{4B^2}\right)|(t_0-t)|}\sin\frac{n\pi x}{L}\cos\frac{\pi}{2}(2m+1)\frac{y}{B} \quad (12)$$

The distribution of φ can be obtained by substitution of equation (12) in equation (9), then we have

$$\varphi(x_0,y_0,t_0)=\left(H_0+\frac{P_0}{\rho}\right)\frac{8}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\left(1+(-1)^{n+1}\right)e^{-\xi_{nm}t_0}}{n(2m+1)}\sin\frac{n\pi x_0}{L}\cos\frac{\pi}{2}(2m+1)\frac{y_0}{B}\cdot$$

$$-\frac{k}{\upsilon}\frac{8}{L^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{n}{(2m+1)}\sin\frac{n\pi x_0}{L}\cos\frac{\pi}{2}(2m+1)\frac{y_0}{B}\cdot\Bigg[\left(H_0+\frac{P_0}{\rho}\right)\times$$
$$\frac{1}{\xi_{nm}}\left(e^{-\xi_{nm}t_0}-e^{-\xi_{nm}|t-t_0|}\right)+\alpha\left\{\frac{-te^{-\xi_{nm}|t-t_0|}}{\xi_{nm}}-\frac{1}{\xi_{nm}^2}\left(e^{-\xi_{nm}t_0}-e^{-\xi_{nm}|t-t_0|}\right)\right\}\Bigg]$$
$$+\frac{k}{\upsilon}\frac{8}{L^2}\left(H_0+\frac{P_0}{\rho}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n+1}n}{(2m+1)\xi_{nm}}\sin\frac{n\pi x_0}{L}\cos\frac{\pi}{2}(2m+1)\frac{y_0}{B}\left(e^{-\xi_{nm}t_0}-e^{-\xi_{nm}|t-t_0|}\right)$$
$$+\frac{k}{\upsilon}\frac{4}{B^2}\frac{P_0}{\rho}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)\left(1+(-1)^{n+1}\right)}{n\ \xi_{nm}}\sin\frac{n\pi x_0}{L}\cos\frac{\pi}{2}(2m+1)\frac{y_0}{B}\left(e^{-\xi_{nm}t_0}-e^{-\xi_{nm}|t-t_0|}\right) \quad (13)$$

where $\xi_{nm} = \frac{k\pi^2}{\nu}\left(\frac{n^2}{L^2} + \frac{(2m+1)^2}{4B^2}\right)$

The capacity is given by

$$c(t) = k\left(H_0 + \frac{P_0}{\rho}\right)\frac{32LB}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(1+(-1)^{n+1}\right)}{(2m+1)^2\xi_{nm}}\left(1-e^{-\xi_{nm}t}\right)$$

$$-\frac{k^2}{\upsilon}\frac{16B}{L^3}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{n^2}{(2m+1)^2}\frac{\left(1+(-1)^{n+1}\right)\left(1-e^{-\xi_{nm}t}\right)}{\xi_{nm}{}^2}\times\left[2\left(H_0+\frac{P_0}{\rho}\right)+\alpha t+\frac{2}{\xi_{nm}}\right]$$

$$+\frac{k^2}{\upsilon}\frac{B^3}{L}\left(H_0+\frac{P_0}{\rho}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n n^2}{(2m+1)^2\xi_{nm}^2}\left(1+(-1)^{n+1}\right)\left(1-e^{-\xi_{nm}t}\right) \quad (14)$$

$$+\frac{k^2}{\upsilon}\frac{32}{BL}\frac{P_0}{\rho}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(1+(-1)^{n+1}\right)}{\xi_{nm}^2}\left(1-e^{-\xi_{nm}t}\right)$$

5. Numerical Results

For a typical example the following numerical values are assumed: $L = 1000\ m$, $B = 100\,m$, $h_0 = 0$ and $\frac{p_0}{\rho} = 10\,m$. ; we assume also that the physical constants $k = 0.01\,m/hr$, $\upsilon = 0.1\,m^{-1}$ and $\alpha = 0.1\,m/hr$. The storage capacity $c(t)$ is calculated and plotted versus the time t as shown in figure. Units of storage capacity is in cubic kilometer and time is measured in hours.

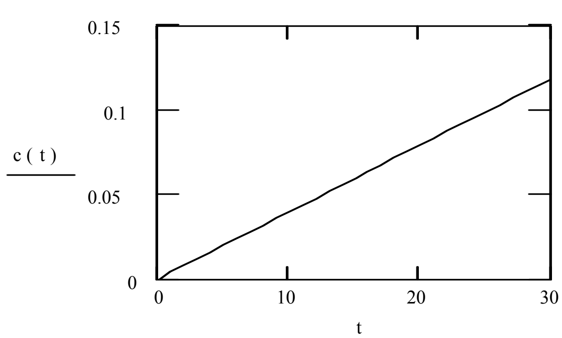

Figure 2: storage capacity

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