# Fuzzy Ideals in CI-algebras

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**Abstract:** The fuzzification of ideals in CI-algebras is considered, and several properties are investigated. Characterizations of a fuzzy ideal are provided. **Mathematical Subject Classification:** 06F35, 03G25, 08A30.
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## 1. Introduction:

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstract: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [6], B. L. Meng introduced the notion of a CI-algebra as a generation of a BE-algebra. In this paper, we consider the fuzzification of ideals in CI-algebras. We introduce the notion of fuzzy ideals in CI-algebra, and investigate related properties. We give characterizations of a fuzzy ideal in CI-algebras.

## 2 Preliminaries

**Definition 2.1 [6]:**

An algebraic system $(X,*,1)$ of type (2, 0) is called a CI-algebra if it satisfying the following conditions:

(1) $x*x=1,$ (2.1)

(2) $1*x=x,$ (2.2)

(3) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$ . (2.3)

We introduce a relation "$\le$" on $X$ by $x\le y$ if and only if $x*y=1$ for all $x,y\in X$ . (2.4)

**Definition 2.2[5]:**

A CI-algebra $(X,*,1)$ is said to be transitive if it satisfies:

$(y*z)*((x*y)*(x*z))=1$ for all $x,y,z\in X$ . (2.5)

A CI-algebra $(X,*,1)$ is said to be self-distributive if it satisfies:

$x*(y*z)=(x*y)*(x*z)$ for all $x,y,z\in X$ . (2.6)

Note that every self-distributive is transitive.

A non-empty subset $S$ of an CI-algebra $X$ is said to be a subalgebra of $X$ if $x*y\in S$ whenever $x,y\in S$ .

In an CI-algebra, the following identities are true:

(4) $y*((y*x)*x)=1$ . (2.7)

(5) $(x*1)*(y*1)=(x*y)*1$ . (2.8)

**Definition 2.3:**

A non empty subset $I$ of a CI-algebra $X$ is said to be a an ideal of $X$ if it satisfies:

($I_1$) If $x\in X$ and $a\in I$ , then $x*a\in I$ , i.e., $X*I\subseteq I$ , (2.9)

($I_2$) If $x\in X$ and $a,b\in I$ , then $(a*(b*x))*x\in I$ . (2.10)

**Lemma 2.4 [5]:**

Let $X$ be a CI-algebra. then

(i) Every ideal of $X$ contains 1,

(ii) If $I$ is an ideal of $X$, then $(a*x)*x\in I$ for all $a\in I$ and $x\in X$ .

## 3 Fuzzy ideals in CI-algebras

In what follows, let $X$ denote a CI-algebra unless otherwise specified.

**Definition 3.1:**

A fuzzy set $\mu$ is called fuzzy ideal of $X$ if it satisfies the following:

1) $\mu(x*y)\ge\mu(y)$ , for all $x,y\in X$ , (3.1)

2) $(\mu((x*(y*z))*z)\ge\min\{\mu(x),\mu(y)\})$ , for all $x,y,z\in X$ . (3.2)

**Theorem 3.2:**
Let $\mu$ be a fuzzy set in *X*. Then $\mu$ is a fuzzy ideal of *X* if and only if it satisfies:
$(\forall \alpha \in [0,1])(U(\mu;\alpha) \neq \phi \Rightarrow U(\mu;\alpha)$ is an ideal of *X*) (3.3)
where $U(\mu;\alpha) := \{x \in X \mid \mu(x) \geq \alpha\}$.

***Proof.*** Assume that $\mu$ is a fuzzy ideal of *X*. Let $\alpha \in [0,1]$ be such that $U(\mu;\alpha) \neq \phi$. Let $x, y \in X$ be such that $y \in U(\mu;\alpha)$. Then $\mu(y) \geq \alpha$, and so $\mu(x * y) \geq \mu(y) \geq \alpha$ by (3.1). Thus $x * y \in U(\mu;\alpha)$. Let $x \in X$ and $a, b \in U(\mu;\alpha)$. Then $\mu(a) \geq \alpha$. It follows from (3.2) that
$\mu((a*(b*x))*x) \geq \min\{\mu(a), \mu(b)\} \geq \alpha$
so that $(a*(b*x))*x \in U(\mu;\alpha)$. Hence $U(\mu;\alpha)$ is an ideal of *X*.
Conversely, suppose that $\mu$ satisfies (3.3). If $\mu(a*b) < \mu(b)$ for some $a, b \in X$, then $\mu(a*b) < \alpha_0 < \mu(b)$ by taking $\alpha_0 := \frac{1}{2}(\mu(a*b) + \mu(b))$. Hence $a*b \notin U(\mu;\alpha_0)$ and $b \in U(\mu;\alpha_0)$, which is a contradiction. Let $a, b, c \in X$ be such that
$\mu((a*(b*c))*c) < \min\{\mu(a), \mu(b)\}$.
Taking
$\beta_0 := \frac{1}{2}(\mu((a*(b*c))*c) + \min\{\mu(a), \mu(b)\}$
, we have $\beta_0 \in [0,1]$ and
$\mu((a*(b*c))*c) < \beta_0 < \min\{\mu(a), \mu(b)\}$.
It follows that $a, b \in U(\mu;\beta_0)$ and $(a*(b*c))*c \notin U(\mu;\beta_0)$. This is a contradiction, and therefore $\mu$ is a fuzzy ideal of *X*.

**Lemma 3.3:**
Every fuzzy ideal $\mu$ of *X* satisfies the following inequality:
$\mu(1) \geq \mu(x)$, for all $x \in X$. (3.4)
***Proof.*** Using (2.1) and (3.1), we have
$\mu(1) = \mu(x*x) \geq \mu(x)$, for all $x \in X$

**Example 3.4:**
Let $X = \{1, a, b, c, d, 0\}$ be a set with the following Cayley table:

| * | 1 | a | b | c | d | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d | 0 |
| a | 1 | 1 | a | c | c | d |
| b | 1 | 1 | 1 | c | c | c |
| c | 1 | a | b | 1 | a | b |
| d | 1 | 1 | a | 1 | 1 | a |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |

Then $(X,*,1)$ is a CI-algebra.
(1) Let $\mu$ be a fuzzy set in *X* defined by
$$\mu(x) := \begin{cases} 0.7 & \text{if } x \in \{1, a, b\}, \\ 0.2 & \text{if } x \in \{c, d, 0\}. \end{cases}$$
Then
$$U\{\mu;\alpha) = \begin{cases} \phi & \text{if } \alpha \in (0.7, 1], \\ \{1, a, b\} & \text{if } \alpha \in (0.2, 0.7], \\ X & \text{if } \alpha \in [0, 0.2]. \end{cases}$$
Note that $\{1, a, b\}$ and *X* are ideals of *X*, and so $\mu$ is a fuzzy ideal of *X*.
(2) Let $\psi$ be a fuzzy set in *X* defined by
$$\psi(x) := \begin{cases} 0.6 & \text{if } x \in \{1, a\}, \\ 0.4 & \text{if } x \in \{b, c, d, 0\}. \end{cases}$$
Then
$$U\{\psi;\beta) = \begin{cases} \phi & \text{if } \beta \in (0.6, 1], \\ \{1, a\} & \text{if } \beta \in (0.4, 0.6], \\ X & \text{if } \beta \in [0, 0.4]. \end{cases}$$
Note that $\{1, a\}$ is not an ideal of *X* since
$(a*(a*b))*b = (a*a)*b = 1*b = b \notin \{1, a\}$
.
Hence $\psi$ is not a fuzzy ideal in *X*.

**Proposition 3.5:**
If $\mu$ is a fuzzy ideal of *X*, then
$(\forall x, y \in X)(\mu((x*y)*y) \geq \mu(x))$ .
(3.5)
***Proof.*** Taking $y = 1$ and $z = y$ in (3.2) and using (2.2) and lemma 3.3, we get
$\mu((x*y)*y) = \mu((x*(1*y))*y) \geq \min\{\mu(x), \mu(1)\} = \mu(x)$

for all $x, y \in X$.

**Corollary 3.6:**

Every fuzzy ideal $\mu$ of $X$ is order preserving, that is, $\mu$ satisfies:

$$(\forall x, y \in X)\ (x \le y \Rightarrow \mu(x) \le \mu(y)). \quad (3.6)$$

***Proof.*** Let $x, y \in X$ be such that $x \le y$. Then $x * y = 1$, and so

$$\mu(y) = \mu(1 * y) = \mu((x * y) * y) \ge \mu(x)$$

by (2.2) and (3.5).

**Proposition 3.7:**

Let $\mu$ be a fuzzy set in $X$ which satisfies (3.4) and

$$(\forall x, y, z \in X)\ (\mu(x * z) \ge \min\{\mu(x * (y * z)), \mu(y)\}). \quad (3.7)$$

Then $\mu$ is order preserving.

***Proof.*** Let $x, y \in X$ be such that $x \le y$. Then $x * y = 1$, and so

$$\mu(y) = \mu(1 * y) \ge \min\{\mu(1 * (x * y)), \mu(x)\} = \min\{\mu(1 * 1), \mu(x)\} = \mu(x)$$

by (2.1), (2.2), (3.7) and (3.4).

We give a characterization of fuzzy ideals.

**Theorem 3.8:**

Let $X$ be a transitive CI-algebra. A fuzzy set $\mu$ in $X$ is a fuzzy ideal of $X$ if it satisfies condition (3.4) and (3.7).

***Proof.*** Assume that $\mu$ is a fuzzy ideal of $X$. By lemma 3.3, $\mu$ satisfies (3.4). Since $X$ is transitive, we have

$$(y * z) * z \le (x * (y * z)) * (x * z) \quad (3.8)$$

i.e., $((y * z) * z) * ((x * (y * z)) * (x * z)) = 1$

for all $x, y, z \in X$. It follows from (2.2), (3.2) and proposition 3.5 that

$$\begin{aligned}\mu(x * z) &= \mu(1 * (x * z)) \\ &= \mu((((y * z) * z) * ((x * (y * z)) * (x * z))) * (x * z)) \\ &\ge \min\{\mu((y * z) * z),\ \mu(x * (y * z))\} \\ &\ge \min\{\mu(x * (y * z)), \mu(y)\}.\end{aligned}$$

Hence $\mu$ satisfies (3.7).

**Corollary 3.9:**

Let $X$ be a self-distributive CI-algebra. A fuzzy set $\mu$ in $X$ is a fuzzy ideal of $X$ if it satisfies condition (3.4) and (3.7).

***Proof.*** Straightforward.

For every $a, b \in X$, let $\mu_a^b$ be a fuzzy set in $X$ defined by

$$\mu_a^b(x) := \begin{cases} \alpha & \text{if } a * (b * x) = 1, \\ \beta & \text{otherwise} \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [0,1]$ with $\alpha > \beta$.

The following example shows that there exist $a, b \in X$ such that $\mu_a^b$ is not a fuzzy ideal of $X$.

**Example 3.10:**

Let $X = \{1, a, b, c, d, 0\}$ be a CI-algebra as in Example 3.4. Then $\mu_1^b$ is not a fuzzy ideal of $X$ since

$$\begin{aligned}\mu_1^a((a * (a * b)) * b) &= \mu_1^a((a * a) * b) = \mu_1^a(1 * b) \\ &= \mu_1^a(b) = \beta < \alpha = \mu_1^a(a) \\ &= \min\{\mu_1^a(a), \mu_1^a(a)\}.\end{aligned}$$

**Lemma 3.11:**

A nonempty subset $I$ of $X$ is an ideal of $X$ if it satisfies

$$1 \in I, \quad (3.9)$$

$$(\forall x, z \in X)(\forall y \in I)(x * (y * z) \in I \Rightarrow x * z \in I). \quad (3.10)$$

***Proof.*** Let $I$ be an ideal of $X$. Using (2.1) and (2.9), we have $1 = a * a \in I$ for all $a \in I$. We prove the following assertion:

$$(\forall x \in I)(\forall y \in X)(x * y \in I \Rightarrow y \in I). \quad (3.11)$$

Let $x \in I$ and $y \in X$ be such that $x * y \in I$. Then

$$y = 1 * y = ((x * y) * (x * y)) * y \in I$$

by (2.10). Now, let $x, z \in X$ and $y \in I$ be such that $x * (y * z) \in I$. Then $y * (x * z) \in I$ by (2.3). Since $y \in I$, it follows from (3.11) that $x * z \in I$. Hence (3.10) is valid.

Let $X$ be an CI-algebra and $a, b \in X$. Define $A(a, b)$ by

$$A(a, b) = \{x \in X \mid a * (b * x) = 1\}.$$

We call $A(a, b)$ an upper set of $a$ and $b$. It is easy to see that $1, a, b \in A(a, b)$ for all $a, b \in X$.

**Theorem 3.12:**

Let $\mu$ be a fuzzy set in $X$. Then $\mu$ is a fuzzy ideal of $X$ if and only if $\mu$ satisfies the following assertion:

$$(\forall a,b\in X)(\forall\alpha\in[0,1])(a,b\in U(\mu;\alpha)\Rightarrow A(a,b)\subseteq U(\mu;\alpha)). \quad (3.12)$$

***Proof.*** Assume that $\mu$ is a fuzzy ideal of *X* and let $a,b\in U(\mu;\alpha)$. Then $\mu(a)\geq\alpha$ and $\mu(b)\geq\alpha$. Let $x\in A(a,b)$. Then $a*(b*x)=1$. Hence

$$\mu(x)=\mu(1*x)=\mu((a*(b*x))*x)\geq\min\{\mu(a),\mu(a)\}\geq\alpha,$$

and so $x\in U(\mu;\alpha)$. Thus $A(a,b)\subseteq U(\mu;\alpha)$. Conversely, suppose that $\mu$ satisfies (3.12). Note that $1\in A(a,b)\subseteq U(\mu;\alpha)$ for all $a,b\in X$. Let $x,y,z\in X$ be such that $x*(y*z)\in U(\mu;\alpha)$ and $y\in U(\mu;\alpha)$. Since

$$(x*(y*z))*(y*(x*z))=(x*(y*z))*(x*(y*z))=1$$

by (2.3) and (2.1), we have $x*z\in A(x*(y*z),y)\subseteq U(\mu;\alpha)$. It follows from Lemma 3.11 that $U(\mu;\alpha)$ is an ideal of *X*. Hence $\mu$ is a fuzzy ideal of *X* by Theorem 3.2.

**Corollary 3.13:**

If $\mu$ is a fuzzy ideal of *X*, then

$$(\forall\alpha\in[0,1])(U(\mu;\alpha)\neq\phi\Rightarrow U(\mu;\alpha)=\bigcup_{a,b\in U(\mu;\alpha)}A(a,b)). \quad (3.12)$$

***Proof.*** Let $\alpha\in[0,1]$ be such that $U(\mu;\alpha)\neq\phi$. Since $1\in U(\mu;\alpha)$, we have

$$U(\mu;\alpha)\subseteq\bigcup_{a\in U(\mu;\alpha)}A(a,1)\subseteq\bigcup_{a,b\in U(\mu;\alpha)}A(a,b).$$

Now, let $x\in\bigcup_{a,b\in U(\mu;\alpha)}A(a,b)$. Then their exist $\nu,\lambda\in U(\mu;\alpha)$ such that $x\in A(\nu,\lambda)\subseteq U(\mu;\alpha)$ by Theorem 3.12. Thus $\bigcup_{a,b\in U(\mu;\alpha)}A(a,b)\subseteq U(\mu;\alpha)$.

This completes the proof.

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**References**

[1] Q. P. Hu and X. Li, On BCH-algebras, Math. Seminar Notes 11(1983), 313-320.
[2] Q. P. Hu and X. Li, On Proper BCH-algebras, Math Japonica 30 (1985), 659-661.
[3] K. Iseki and S. Tanaka, An introduction to theory of BCK-algebras, Math Japonica 23 (1978), 1-20.
[4] K. Iseki, On BCI-algebras, Math. Seminar Notes 8 (1980), 125-130.
[5] K. H. Kim, A Note on CI-algebras, International Mathematical Forum, 6 (2011), 1-5.
[6] B. L. Meng, CI-algebras, Sci. Math. Japo. Online, e-2009, 695-701.

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