Fuzzy TM-ideals of TM-algebras

Samy M. Mostafa[1], Mokhtar A. Abdel Naby[2] and Osama R. Elgendy[3]

Department of Mathematics, Faculty of Eduction, Ain Shams University, Roxy, Cairo, Egypt
dr_usamaelgendy@yahoo.com

Abstract: The fuzzification of TM- ideals in TM-algebras is considered, and several properties are investigated. Characterizations of a fuzzy ideal are provided.
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1. Introduction:

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstract: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [6], J. Neggers, S. S. Ahn and H. S. Kim introduced Q-algebras which is a generalization of BCK / BCI-algebras and obtained several results. In [5], K. Megalai and A. Tamilarasi introduced a class of abstract algebras: TM-algebras which is a generalization of Q / BCK / BCI / BCH-algebras. In this paper, we consider the fuzzification of TM-ideals in TM-algebras. We introduce the notion of fuzzy TM-ideals in CI-algebras, and investigate related properties. We investigate how to deal with the homomorphic and inverse image of fuzzy TM-ideals in TM-algebras.

2 Preliminaries

In this section, certain definitions, Known results and examples that will be used in the sequel are described.

Definition 2.1:

A BCI-algebra is an algebra $(X,*,0)$ of type (2,0) satisfying the following conditions:

i) $(x*y)*(x*z) \le z*y$

ii) $x*(x*y) \le y$

iii) $x \le x$

iv) $x \le y$ and $y \le x$ imply $x = y$

v) $x \le 0$ implies $x = 0$, where $x \le y$ is defined by $x*y = 0$ for all $x, y, z \in X$.

Definition 2.2:

A BCK-algebra is an algebra $(X,*,0)$ of type (2,0) satisfying the following conditions:

i) $(x*y)*(x*z) \le z*y$

ii) $x*(x*y) \le y$

iii) $x \le x$

iv) $x \le y$ and $y \le x$ imply $x = y$

v) $0 \le x$ implies $x = 0$, where $x \le y$ is defined by $x*y = 0$ for all $x, y, z \in X$.

Definition 2.3:

A BCH-algebra is an algebra $(X,*,0)$ of type (2,0) satisfying the following conditions:

i) $x*x = 0$

ii) $(x*y)*z = (x*z)*y$

iii) $x*y = 0$ and $y*x = 0$ imply $x = y$ for all $x, y, z \in X$.

Definition 2.4:

A Q-algebra is an algebra $(X,*,0)$ of type (2,0) satisfying the following condition:

i) $x*x = 0$

ii) $x*0 = x$

iii) $(x*y)*z = (x*z)*y$, for all $x, y, z \in X$.

Every BCK-algebra is a BCI-algebra but not conversely.

Every BCI-algebra is a BCH-algebra but not conversely.

Every BCH-algebra is a Q-algebra but not conversely.

Every Q-algebra satisfying the conditions $(x*y)*(x*z) = z*y$ and $x*y = 0$ and $y*x = 0$ imply $x = y$ is a BCI-algebra.

Definition 2.5 (TM-algebra):

A TM-algebra is an algebra $(X,*,0)$ is a non empty subset X with a constant "0" and a binary operation "$*$" satisfying the following axioms:
i) $x*0=x$
ii) $(x*y)*(x*z)=z*y$, for all $x,y,z\in X$.
In X we can define a binary operation $\leq$ by $x\leq y$ if and only if $x*y=0$.

In any TM-algebra $(X,*,0)$, the following holds good for all $x,y,z\in X$
a) $x*x=0$,
b) $(x*y)*x=0*y$,
c) $x*(x*y)=y$,
d) $(x*z)*(y*z)\leq x*y$,
e) $(x*y)*z=(x*z)*y$,
f) $x*0=0\Rightarrow x=0$,
h) $x*z\leq y*z$ and $z*y\leq z*x$,
i) $x*(x*(x*y))=x*y$,
j) $0*(x*y)=y*x=(0*x)*(0*y)$,
k) $(x*(x*y))*y=0$,
l) $x*y=0$ and $y*x=0$ imply $x=y$.

A QS-algebra is obviously a TM-algebra, but a TM-algebra is said to be QS-algebra if it satisfies the additional relations $(x*y)*z=(x*z)*y$ and $y*z=z*y$ for all $x,y,z\in X$.

Example 2.6:

Let $X=\{0,1,2,3\}$ be a set with a binary operation $*$ defined by the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	0	0	0

Then $(X,*,0)$ is a TM-algebra.

Definition 2.7:

A non empty subset I of a BCK-algebra X is said to be a BCK-ideal of X if it satisfies:
(I_1) $0\in I$,
(I_2) $x*y\in I$ and $y\in I$ implies $x\in I$ for all $x,y\in X$.

Definition 2.8(TM-ideal):

Let $(X,*,0)$ be a TM-algebra. A non-empty subset I of X is called TM- ideal of X if it satisfies the following conditions:
(I_1) $0\in I$,
(T_2) $x*z\in I$ and $z*y\in I$ imply $x*y\in I$, for all $x,y,z\in X$.

Definition 2.9:

A non empty subset S of a TM-algebra X is said to be TM-subalgebra of X, if $x,y\in S$, implies $x*y\in S$.

Proposition 2.10:

Let $(X,*,0)$ be a TM-algebra and I is a TM-ideal of X, then I is a BCK-ideal of X.

Proof. I_1 is satisfied.
Put in (T_2) $y=0$, we have $x*z\in I$ and $z*0=z\in I$ imply $x*0=x\in I$, for all x,y and $z\in X$ i.e. I is a BCK-ideal of X.

Example 2.11:

Let $X=\{0,1,2,3\}$ as in example 2.6, and $A=\{0,1,2\}$ is a TM-ideal of TM-algebra X.

3 Homomorphism of TM-algebras:

Let $(X,*,0)$ and $(Y,*,0)$ be a TM-algebras. A mapping $f:X\to Y$ is called a homomorphism if $f(x*y)=f(x)*f(y)$, for all $x,y\in X$. A homomorphism f is called monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two TM-algebras X and Y are said to be isomorphic, written by $X\cong Y$, if there exist isomorphism $f:X\to Y$. For any homomorphism $f:X\to Y$, the set $\{x\in X\mid f(x)=0\}$ is called the kernel of f, denoted by $\ker(f)$ and the set $\{f(x)\mid x\in X\}$ is called the image of f, denoted by $\operatorname{Im}(f)$. We denoted by $Hom(X,Y)$ the set of all homomorphisms of TM-algebras from X to Y.

Proposition 3.1:

Let $(X,*,0)$ and $(Y,*,0)$ be a TM-algebras. A mapping $f:X\to Y$ is homomorphism of TM-algebras, then the $\ker(f)$ is TM-ideal.

Proof. Let $x*z\in\ker(f)$ and $z*y\in\ker(f)$ then

$f(x*z)=0'$ and $f(z*y)=0'$.

Since

$0'=f(z*y)=f((x*y)*(x*z))=f(x*y)*'f(x*z)$

$0'=f(x*y)*'0'$ by using (definition 2.5),

$0'=f(x*y)$, hence $x*y\in\ker f$.

4 Fuzzy TM-ideals of TM-algebras:

Definition 4.1:

Let *X* be a set. A fuzzy set μ in *X* is a function $\mu: X\to[0,1]$.

Definition 4.2[6]:

Let *X* be a BCK-algebra. a fuzzy set μ in *X* is called a fuzzy BCK-ideal of *X* if it satisfies:

(FI_1) $\mu(0)\ge\mu(x)$,

(FI_2) $\mu(x)\ge\min\{\mu(x*y),\mu(y)\}$, for all x, y and $z\in X$.

Definition 4.3:

Let *X* be a TM-algebra. A fuzzy set μ in *X* is called a fuzzy TM-ideal of *X* if it satisfies:

(FI_1) $\mu(0)\ge\mu(x)$,

(FT) $\mu(x*y)\ge\min\{\mu(x*z),\mu(z*y)\}$, for all $x,y,z\in X$.

Example 4.4:

Let $X=\{0,1,2,3,4\}$ as in example 2.6, and let $t_0,t_1,t_2\in[0,1]$ be such that $t_0>t_1>t_2$. Define a mapping $\mu: X\to[0,1]$ by $\mu(0)=t_0$, $\mu(1)=t_1$, $\mu(2)=\mu(3)=t_2$.

Routine calculations give that μ is a fuzzy TM-ideal of *X*.

Theorem 4.5:

Any fuzzy TM-ideal of TM-algebra *X* is fuzzy BCK-ideal of *X*.

Proof. (FI_1) is satisfied.

Put $y=0$ in (FT), we have

$$\mu(x*0)=\mu(x)\ge\min\{\mu(x*z),\mu(z*0)\}$$
$$=\min\{\mu(x*z),\mu(z)\},$$

hence we obtain (FI_2).

Lemma 4.6:

If μ is a fuzzy TM-ideal of TM-algebra *X*, then $x\le z$ implies $\mu(x)\ge\mu(z)$.

Proof. If $x\le z$ then $x*z=0$, this together with $x*0=x$ and $\mu(0)\ge\mu(x)$, gives

$$\mu(x*0)=\mu(x)\ge\min\{\mu(x*z),\mu(z*0)\}$$
$$\ge\min\{\mu(0),\mu(z)\}$$
$$\ge\mu(z).$$

Theorem 4.7:

The intersection of any set of fuzzy TM-ideal in TM-algebra *X* is also a fuzzy TM-ideal.

Proof. Let $\{\mu_i\}$ be a family of fuzzy TM-ideals of TM-algebras *X*.

Then for any $x,y,z\in X$,

$(\bigcap\mu_i)(0)=\inf(\mu_i(0))\ge\inf(\mu_i(x))=(\bigcap\mu_i)(x)$, and

$$(\bigcap\mu_i)(x*y)=\inf(\mu_i(x*y))$$
$$\ge\inf(\min\{\mu_i(x*z),\mu_i(z*y))$$
$$=\min\{\inf(\mu_i(x*z)),\inf(\mu_i(z*y))\}$$
$$=\min\{(\bigcap\mu_i)(x*z),(\bigcap\mu_i)(z*y)\}.$$

This completes the proof.

Theorem 4.8:

Let *A* be a non-empty subset of a TM-algebra *X* and μ be a fuzzy subset of *X* such that μ is into $\{0,1\}$, so that μ is the characteristic function of *A*. Then μ is a fuzzy TM-ideal of *X* if and only if *A* is a TM-ideal of *X*.

Proof. Assume that μ is a fuzzy TM-ideal of *X*. Since $\mu(0)\ge\mu(x)$ for all $x\in X$, clearly we have $\mu(0)=1$, and so $0\in A$. Let $x*z\in A$ and $z*y\in A$. Since μ is a fuzzy TM-ideal of *X*, it follows that $\mu(x*y)\ge\min\{\mu(x*z),\mu(z*y)\}=1$, and that $\mu(x*y)=1$.

This means that $\mu(x*y)\in A$, i.e., *A* is TM-ideal of *X*.

Conversely suppose *A* is a TM-ideal of *X*. Since $0\in A$, $\mu(0)=1\ge\mu(x)$ for all $x\in X$. Let $x,y,z\in X$. If $z*y\notin A$, then $\mu(z*y)=0$, and so $\mu(x*y)\ge0=\min\{\mu(x*z),\mu(z*y)\}$, if $x*y\notin A$, and $z*y\in A$, then $x*z\notin A$ (*A* is TM-ideal).

Thus $\mu(x*y)=0=\min\{\mu(x*z),\mu(z*y)\}$, therefore μ is a fuzzy TM-ideal of *X*.

Definition 4.9:

Let *f* be a mapping from the set *X* to a set *Y*. If μ is a fuzzy subset of *X*, then the fuzzy subset *B* of *Y* defined by

$$f(\mu)(y) = B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

Is called the image of μ under *f*.

Similarly, if *B* is a fuzzy subset of *Y*, then the fuzzy subset defined by $\mu(x) = B(f(x))$ for all $x \in X$, is said to be the preimage of *B* under *f*.

Theorem 4.10:

An into homomorphic preimage of a fuzzy TM-ideal is also fuzzy TM-ideal.

Proof. Let $f : X \to X'$ be an into homomorphism of TM-algebras, *B* a fuzzy TM-ideal of X' and μ the preimage of *B* under f. Then $B(f(x)) = \mu(x)$, for all $x \in X$, (FI_1) hold, since $\mu(0) = B(f(0)) \geq B(f(x)) = \mu(x)$. Let $x, y, z \in X$, then

$$\begin{aligned}\mu(x * y) &= B(f(x * y)) = B(f(x) *' f(y)) \\ &\geq \min\{B(f(x) * f(z)), B(f(z) * f(y))\} \\ &= \min\{B(f(x * z)), B(f(z * y))\} \\ &= \min\{\mu(x * z), \mu(z * y)\}.\end{aligned}$$

Hence $\mu(x) = B(f(x)) = (B \circ f)(x)$ is a fuzzy TM-ideal of *X*. The proof is completed.

Theorem 4.11:

Let $f : X \to Y$ be a homomorphism between TM-algebras *X* and *Y*.

For every fuzzy TM-ideal μ in *X*, $f(\mu)$ is a fuzzy TM-ideal of *Y*.

Proof. By definition $B(y') = f(\mu)(y') := \sup\limits_{x \in f^{-1}(y')} \mu(x)$ for all $y' \in Y$ and $\sup \phi := 0$

We have to prove that

$B(x' * y') \geq \min\{B(x' * z'), B(z' * y')\}$, for all $x', y', z' \in Y$.

(i) Let $f : X \to Y$ be an onto homomorphism of TM-algebras. Let μ be a fuzzy TM-ideal of *X* with sup property and *B* the image of μ under *f*. Since μ is a fuzzy TM-ideal of *X*, we have $\mu(0) \geq \mu(x)$, for all $x \in X$. Note that $0 \in f^{-1}(0')$, where 0 and $0'$ are the zeroes elements of *X* and *Y* respectively.

Thus, $B(0') = \sup\limits_{t \in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x)$, for all $x \in X$, which implies that $B(0') = \sup\limits_{t \in f^{-1}(x')} \mu(t) = B(x')$, for any $x' \in Y$.

For any $x', y', z' \in Y$, let $x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y'), z_0 \in f^{-1}(z')$ be such that

$\mu(x_0) = \sup\limits_{t \in f^{-1}(x')} \mu(t)$, $\mu(y_0) = \sup\limits_{t \in f^{-1}(y')} \mu(t)$

and $\mu(z_0) = \sup\limits_{t \in f^{-1}(z')} \mu(t)$

and

$$\mu(x_0 * z_0) = B\{f(x_0 * z_0)\} = B(x' * z') = \sup_{(x_0 * z_0) \in f^{-1}(x' * z')} \{\mu(x_0 * z_0)\}$$

$$= \sup_{t \in f^{-1}(x' * z')} \mu(t).$$

Then

$$B(x' * y') = \sup_{t \in f^{-1}(x' * y')} \mu(t) = \mu(x_0 * y_0)$$

$$\geq \min\{\mu(x_0 * z_0), \mu(z_0 * y_0)\} =$$

$$\min\left\{\sup_{t \in f^{-1}(x' * z')} \mu(t), \sup_{t \in f^{-1}(z' * y')} \mu(t)\right\} =$$

$$\min\{B(x' * z'), B(z' * y')\}.$$

Hence *B* is a fuzzy TM-ideal of *Y*.

(ii) If f is not onto. For every $x' \in Y$ we define $X_{x'} := f^{-1}(x')$. Since *f* is a homomorphism we have $(X_{x'} * X_{z'}) \subset X_{(x' * z')}$ for all $x', y', z' \in Y$ ....(v).

Let $x', y', z' \in Y$ be an arbitrary given. If $(x' * z') \notin \operatorname{Im}(f) = f(X)$, then by definition $B(x' * z') = 0$. But if $(x' * z') \notin f(X)$ .i.e. $X_{(x' * z')} = \phi$, then by (v) at least one of x', y' and $z' \notin f(X)$, and hence $B(x' * y') \geq 0 = \min\{B(x' * z'), B(z' * y')\}$.

Corresponding author

Osama R. Elgendy, Department of mathematics, Ain Shams University, Roxy, Cairo, Egypt.

dr_usamaelgendy@yahoo.com

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