Fuzzy TM-ideals of TM-algebras

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Abstract: The fuzzification of TM-ideals in TM-algebras is considered, and several properties are investigated. Characterizations of a fuzzy ideal are provided.

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1. Introduction: Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstract: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [6], J. Neggers, S. S. Ahn and H. S. Kim introduced Q-algebras which is a generalization of BCK / BCI-algebras and obtained several results. In [5], K. Megalai and A. Tamilarasi introduced a class of abstract algebras: TM-algebras which is a generalization of Q / BCK / BCI / BCH-algebras. In this paper, we consider the fuzzification of TM-ideals in TM-algebras. We introduce the notion of fuzzy TM-ideals in CI-algebras, and investigate related properties. We investigate how to deal with the homomorphic and inverse image of fuzzy TM-ideals in TM-algebras.

2 Preliminaries

In this section, certain definitions, Known results and examples that will be used in the sequel are described.

Definition 2.1: A BCI-algebra is an algebra \((X,*,0)\) of type (2,0) satisfying the following conditions:

i) \((x*y)*(x*z) \leq z*y\)

ii) \(x*(x*y) \leq y\)

iii) \(x \leq x\)

iv) \(x \leq y\) and \(y \leq x\) imply \(x = y\)

v) \(0 \leq x\) implies \(x = 0\), where \(x \leq y\) is defined by \(x*y = 0\) for all \(x, y, z \in X\).

Definition 2.2: A BCK-algebra is an algebra \((X,*,0)\) of type (2,0) satisfying the following conditions:

i) \((x*y)*(x*z) \leq z*y\)

ii) \(x*(x*y) \leq y\)

iii) \(x \leq x\)

iv) \(x \leq y\) and \(y \leq x\) imply \(x = y\)

v) \(0 \leq x\) implies \(x = 0\), where \(x \leq y\) is defined by \(x*y = 0\) for all \(x, y, z \in X\).

Definition 2.3: A BCH-algebra is an algebra \((X,*,0)\) of type (2,0) satisfying the following conditions:

i) \(x*x = 0\)

ii) \((x*y)*z = (x*z)*y\)

iii) \(x*y = 0\) and \(y*x = 0\) imply \(x = y\) for all \(x, y, z \in X\).

Definition 2.4: A Q-algebra is an algebra \((X,*,0)\) of type (2,0) satisfying the following condition:

i) \(x*x = 0\)

ii) \(x*0 = x\)

iii) \((x*y)*(x*z) = (x*z)*y\) for all \(x, y, z \in X\).

Every BCK-algebra is a BCI-algebra but not conversely.

Every BCI-algebra is a BCH-algebra but not conversely.

Every BCH-algebra is a Q-algebra but not conversely.

Every Q-algebra satisfying the conditions \((x*y)*(x*z) = z*y\) and \(x*y = 0\) and \(y*x = 0\) imply \(x = y\) is a BCI-algebra.

Definition 2.5 (TM-algebra):
A TM-algebra is an algebra \((X, *, 0)\) is a non empty subset \(X\) with a constant “0” and a binary operation “*” satisfying the following axioms:

i) \(x * x = 0\),

ii) \((x * y) * (x * z) = z * y\), for all \(x, y, z \in X\).

In \(X\) we can define a binary operation \(\leq\) by \(x \leq y\) if and only if \(x * y = 0\).

In any TM-algebra \((X, *, 0)\), the following holds good for all \(x, y, z \in X\):

a) \(x * x = 0\),

b) \((x * y) * x = 0 * y\),

c) \(x * (x * y) = y\),

d) \((x * z) * (y * z) \leq x * y\),

e) \((x * y) * z = (x * z) * y\),

f) \(x * 0 = 0 = 0 * x\),

h) \(x * z \leq y * z\) and \(z * y \leq z * x\),

i) \(x * (x * (x * y)) = x * y\),

j) \(0 * (x * y) = y * x = (0 * x) * (0 * y)\),

k) \((x * (x * y)) * y = 0\),

l) \(x * y = 0\) and \(y * x = 0\) imply \(x = y\).

Example 2.6:

Let \(X = \{0, 1, 2, 3\}\) be a set with a binary operation * defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X, *, 0)\) is a TM-algebra.

Definition 2.7:

A non empty subset \(I\) of a BCK-algebra \(X\) is said to be a BCK-ideal of \(X\) if it satisfies:

(i) \(0 \in I\),

(ii) \(x * y \in I\) and \(y \in I\) implies \(x \in I\) for all \(x, y \in X\).

Definition 2.8(TM-ideal):

Let \((X, *, 0)\) be a TM-algebra. A non-empty subset \(I\) of \(X\) is called TM-ideal of \(X\) if it satisfies the following conditions:

\(\text{(i)}\)

\[0 \in I,\]

\(\text{(ii)}\)

\[x * z \in I \quad \text{and} \quad z * y \in I \implies x * y \in I, \quad \text{for all} \quad x, y, z \in X.\]

Definition 2.9:

A non empty subset \(S\) of a TM-algebra \(X\) is said to be TM-subalgebra of \(X\), if \(x, y \in S\), implies \(x * y \in S\).

Proposition 2.10:

Let \((X, *, 0)\) be a TM-algebra and \(I\) is a TM-ideal of \(X\), then \(I\) is a BCK-ideal of \(X\).

Proof. \(I_1\) is satisfied.

Put \(y = 0\), we have \(x * z \in I\) and \(z * 0 = z \in I\) imply \(x * 0 = x \in I\), for all \(x, y, z \in X\) i.e. \(I\) is a BCK-ideal of \(X\).

Example 2.11:

Let \(X = \{0, 1, 2, 3\}\) as in example 2.6, and \(A = \{0, 1, 2\}\) is a TM-ideal of TM-algebra \(X\).

3 Homomorphism of TM-algebras:

Let \((X, *, 0)\) and \((Y, *, 0)\) be a TM-algebra. A mapping \(f: X \rightarrow Y\) is called a homomorphism if \(f(x * y) = f(x) * f(y)\), for all \(x, y \in X\). A homomorphism \(f\) is called monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two TM-algebras \(X\) and \(Y\) are said to be isomorphic, written by \(X \cong Y\), if there exist isomorphism \(f: X \rightarrow Y\). For any homomorphism \(f: X \rightarrow Y\), the set \(\{x \in X \mid f(x) = 0\}\) is called the kernel of \(f\), denoted by \(\ker(f)\) and the set \(\{f(x) \mid x \in X\}\) is called the image of \(f\), denoted by \(\text{Im}(f)\). We denoted by \(\text{Hom}(X, Y)\) the set of all homomorphisms of TM-algebras from \(X\) to \(Y\).

Proposition 3.1:

Let \((X, *, 0)\) and \((Y, *, 0)\) be a TM-algebra. A mapping \(f: X \rightarrow Y\) is homomorphism of TM-algebras, then the \(\ker(f)\) is TM-ideal.

Proof. Let \(x * z \in \ker(f)\) and \(z * y \in \ker(f)\) then
\[
f(x \ast z) = 0'
\] and \[
f(z \ast y) = 0'.
\]
Since
\[
0' = f(z \ast y) = f((x \ast y) \ast (x \ast z)) = f(x \ast y) \ast f(x \ast z)
\]
\[
0' = f(x \ast y) \ast 0' \quad \text{by using (definition 2.5)},
\]
\[
0' = f(x \ast y), \text{ hence } x \ast y \in \ker f.
\]

4 Fuzzy TM-ideals of TM-algebras:

Definition 4.1:
Let \( X \) be a set. A fuzzy set \( \mu \) in \( X \) is a function
\[
\mu : X \rightarrow [0,1].
\]

Definition 4.2[6]:
Let \( X \) be a BCK-algebra. A fuzzy set \( \mu \) in \( X \) is called a fuzzy BCK-ideal of \( X \) if it satisfies:
\[
\text{(Fl)} \quad \mu(0) \geq \mu(x),
\]
\[
\text{(FT)} \quad \mu(x \ast y) \geq \min\{\mu(x \ast z), \mu(z \ast y)\}, \text{ for all } x, y, z \in X.
\]

Definition 4.3:
Let \( X \) be a TM-algebra. A fuzzy set \( \mu \) in \( X \) is called a fuzzy TM-ideal of \( X \) if it satisfies:
\[
\text{(Fl)} \quad \mu(0) \geq \mu(x),
\]
\[
\text{(FT)} \quad \mu(x \ast y) \geq \min\{\mu(x \ast z), \mu(z \ast y)\}, \text{ for all } x, y, z \in X.
\]

Example 4.4:
Let \( X = \{0,1,2,3,4\} \) as in example 2.6, and let \( t_0, t_1, t_2 \in [0,1] \) be such that \( t_0 > t_1 > t_2 \). Define a mapping
\[
\mu : X \rightarrow [0,1]
\]
by
\[
\mu(0) = t_0 , \quad \mu(1) = t_1 , \quad \mu(2) = \mu(3) = t_2.
\]
Routine calculations give that \( \mu \) is a fuzzy TM-ideal of \( X \).

Theorem 4.5:
Any fuzzy TM-ideal of TM-algebra \( X \) is fuzzy BCK-ideal of \( X \).

Proof. (Fl) is satisfied.

Let \( y = 0 \) in (FT), we have
\[
\mu(x \ast 0) = \mu(x) \geq \min\{\mu(x \ast z), \mu(z \ast 0)\}
\]
\[
= \min\{\mu(x \ast z), \mu(z)\},
\]
hence we obtain (Fl).

Lemma 4.6:
If \( \mu \) is a fuzzy TM-ideal of TM-algebra \( X \), then \( x \leq z \) implies \( \mu(x) \geq \mu(z) \).

Proof. If \( x \leq z \) then \( x \ast z = 0 \), this together with \( x \ast 0 = x \) and \( \mu(0) \geq \mu(x) \), gives
\[
\mu(x \ast 0) = \mu(x) \geq \min\{\mu(x \ast z), \mu(z \ast 0)\}
\]
\[
\geq \min\{\mu(0), \mu(z)\}
\]
\[
\geq \mu(z).
\]

Theorem 4.7:
The intersection of any set of fuzzy TM-ideal in TM-algebra \( X \) is also a fuzzy TM-ideal.

Proof. Let \( \{\mu_i\} \) be a family of fuzzy TM-ideals of TM-algebra \( X \). Then for any \( x, y, z \in X \),
\[
(\bigcap \mu_i)(0) = \inf(\mu_i(0)) \geq \inf(\mu_i(x)) = (\bigcap \mu_i)(x)
\]
and
\[
(\bigcap \mu_i)(x \ast y) = \inf(\mu_i(x \ast y))
\]
\[
\geq \inf(\min\{\mu_i(x \ast z), \mu_i(z \ast y)\})
\]
\[
= \min(\inf(\mu_i(x \ast z)), \inf(\mu_i(z \ast y)))
\]
\[
= \min((\bigcap \mu_i)(x \ast z), (\bigcap \mu_i)(z \ast y)).
\]
This completes the proof.

Theorem 4.8:
Let \( A \) be a non-empty subset of a TM-algebra \( X \) and \( \mu \) be a fuzzy subset of \( X \) such that \( \mu \) is into \( \{0,1\} \), so that \( \mu \) is the characteristic function of \( A \). Then \( \mu \) is a fuzzy TM-ideal of \( X \) if and only if \( A \) is a TM-ideal of \( X \).

Proof. Assume that \( \mu \) is a fuzzy TM-ideal of \( X \). Since \( \mu(0) \geq \mu(x) \) for all \( x \in X \), clearly we have \( \mu(0) = 1 \), and so \( 0 \in A \). Let \( x \ast y \in A \) and \( z \ast y \in A \). Since \( \mu \) is a fuzzy TM-ideal of \( X \), it follows that \( \mu(x \ast y) \geq \min\{\mu(x \ast z), \mu(z \ast y)\} = 1 \), and that \( \mu(x \ast y) = 1 \).
This means that \( \mu(x \ast y) \in A \), i.e., \( A \) is TM-ideal of \( X \). Conversely suppose \( A \) is a TM-ideal of \( X \). Since \( 0 \in A \) , \( \mu(0) = 1 \geq \mu(x) \) for all \( x \in X \). Let \( x, y, z \in X \). If \( z \ast y \in A \), then \( \mu(z \ast y) = 0 \), and so \( \mu(x \ast y) \geq 0 = \min\{\mu(x \ast z), \mu(z \ast y)\} \), if \( x \ast y \in A \), and \( z \ast y \in A \), then \( x \ast z \notin A \) (\( A \) is TM-ideal). Thus \( \mu(x \ast y) = 0 = \min\{\mu(x \ast z), \mu(x \ast y)\} \), therefore \( \mu \) is a fuzzy TM-ideal of \( X \).
Let \( f \) be a mapping from the set \( X \) to a set \( Y \). If \( \mu \) is a fuzzy subset of \( X \), then the fuzzy subset \( B \) of \( Y \) defined by
\[
f(\mu)(y) = B(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}
\]
is called the image of \( \mu \) under \( f \).

Similarly, if \( B \) is a fuzzy subset of \( Y \), then the fuzzy subset defined by \( \mu(x) = B(f(x)) \) for all \( x \in X \), is said to be the preimage of \( B \) under \( f \).

Let \( f : X \rightarrow X' \) be an into homomorphism of TM-algebras, \( B \) a fuzzy TM-ideal of \( X' \) and \( \mu \) the preimage of \( B \) under \( f \). Then \( B(f(x)) = \mu(x) \), for all \( x \in X \), (F1) hold, since \( \mu(0) = B(f(0)) \geq B(f(x)) = \mu(x) \).

Hence \( \mu(x) = B(f(x)) = (B \circ f)(x) \) is a fuzzy TM-ideal of \( X \). The proof is completed.

Let \( f : X \rightarrow Y \) be a homomorphism between TM-algebras \( X \) and \( Y \).

For every fuzzy TM-ideal \( \mu \) in \( X \), \( f(\mu) \) is a fuzzy TM-ideal of \( Y \).

By definition \( B'(y') = f(\mu)(y') = \sup_{x \in f^{-1}(y')} \mu(x) \) for all \( y' \in Y \) and sup \( \mu = 0 \)
all \( x \in X \) and sup \( \mu = 0 \)
We have to prove that
\[
B(x' \ast y') \geq \min \{B(x' \ast z'), B(z' \ast y')\}, \quad \text{for all } x', y', z' \in Y.
\]
(i) Let \( f : X \rightarrow Y \) be an onto homomorphism of TM-algebras. Let \( \mu \) be a fuzzy TM-ideal of \( X \) with sup property and \( B \) the image of \( \mu \) under \( f \). Since \( \mu \) is a fuzzy TM-ideal of \( X \), we have \( \mu(0) \geq \mu(x) \), for all \( x \in X \). Note that \( 0 \in f^{-1}(0') \), where 0 and 0' are the zeroes elements of \( X \) and \( Y \) respectively.

Thus, \( B(0') = \sup_{t \in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x) \), for all \( x \in X \), which implies that
\( B(0') = \sup_{t \in f^{-1}(x')} \mu(t) = B(x') \), for any \( x' \in Y \).

For any \( x', y', z' \in Y \), let \( x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y'), z_0 \in f^{-1}(z') \) be such that
\( \mu(x_0) = \sup_{t \in f^{-1}(x')} \mu(t), \mu(y_0) = \sup_{t \in f^{-1}(y')} \mu(t) \)
and \( \mu(z_0) = \sup_{t \in f^{-1}(z')} \mu(t) \)

and \( \mu(x_0 \ast z_0) = B(f(x_0 \ast z_0)) = B(x' \ast z') = \sup_{(t_0,t_0) \in f^{-1}(x' \ast z')} \{\mu(t_0)\} \)

Thus, \( B(x' \ast y') = \sup_{t \in f^{-1}(x' \ast y')} \mu(t) = \mu(x_0 \ast y_0) \)
\( \geq \min \{\mu(x_0 \ast z_0), \mu(z_0 \ast y_0)\} = \min \left\{ \sup_{t \in f^{-1}(x' \ast z')} \mu(t), \sup_{t \in f^{-1}(z' \ast y')} \mu(t) \right\} \)
\[ = \min \{B(x' \ast z'), B(z' \ast y')\}. \]

Hence \( B \) is a fuzzy TM-ideal of \( Y \).

(ii) If \( f \) is not onto. For every \( x' \in Y \) we define \( X' := f^{-1}(x') \). Since \( f \) is a homomorphism we have \( (X' \ast X') \subseteq X_{(x' \ast x')} \) for all \( x', y', z' \in Y \).

Let \( x', y', z' \in Y \) be an arbitrary given. If \( (x' \ast z') \in \text{Im}(f) = f(X) \), then by definition
\( B(x' \ast z') = 0 \). But if \( (x' \ast z') \notin f(X) \), i.e. \( X_{(x' \ast z')} = \emptyset \), then by (v) at least one of \( x', y' \) and \( z' \notin f(X) \), and hence
\( B(x' \ast y') \geq 0 = \min \{B(x' \ast z'), B(z' \ast y')\} \).

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References


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