Anti-Fuzzy Sub-Implicative Ideals of BCI-Algebras

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Abstract: In this paper, we introduce the notion of anti-fuzzy sub-implicative ideal of BCI-algebras, and study some of their properties. We show that a fuzzy subset of BCI-algebra is a fuzzy sub-implicative ideal if and only if the complement of this fuzzy subset is an anti-fuzzy sub-implicative ideal, and any anti- fuzzy ideal of implicative BCI-algebra is anti-fuzzy sub-implicative ideal. We investigate how to deal with the homomorphic image (pre-image) of anti-fuzzy sub-implicative ideal of BCI-algebra. Moreover, we introduce the notion of Cartesian product of anti-fuzzy sub-implicative ideals and then we study some related properties.

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1. Introduction

The concept of a fuzzy set was introduced by Zadeh [17] and was used afterwards by many other outhers in various branches of mathematics. In 1966, Imai and Ise′ki [6] introduced the notion of BCI-algebras. Xi [16] applied the concept of fuzzy set to BCI-algebras and gave some properties of it. After that Jun and Meng investigated further properties of fuzzy BCI-algebras and fuzzy ideal [see {[2] , [13] , [7] , [8] , [10]}]. S .M .Mostafa [15] gave some properties of a fuzzy implicative ideal in BCK-algebra .Liu and Meng [11] introduced the notion of sub-implicative ideal and sub-commutative ideal in BCI-algebra and investigated the properties of this ideals. [2] Biswas introduced the concept of anti-fuzzy sub-group. Modifying this idea, in this paper, we introduce the concept of anti-fuzzy sub implicative ideal of BCI-algebra and investigate some related properties. We show that in implicative BCI-algebra a fuzzy subset is an anti-fuzzy ideal if and only if it is anti-fuzzy sub-implicative ideal, and a fuzzy subset of a BCI-algebra is a fuzzy sub-implicative ideal if and only if the complement of this fuzzy subset is an anti-fuzzy sub implicative ideal. Moreover, we discuss the homomorphic pre-image (image) of anti-fuzzy sub-implicative ideal. Finally, we introduce the notion of Cartesian product of anti-fuzzy sub-implicative ideal and then we characterize anti-fuzzy sub-implicative ideal by it.

2. Preliminaries

Definition 2.1. ([6])

An algebra $(X; *, 0)$ of type $(2,0)$ is called a BCI-algebra if it satisfies the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y, z \in X$.

We can define a partially ordered relation $\leq$ on X as follows:

$x \leq y$ if and only if $x * y = 0$.

Proposition 2.2. ([6])

A BCI-algebra X satisfies the following properties:

(1) $(x * y) * z = (x * z) * y$,

(2) $x * 0 = x$,

(3) $0 * (x * y) = (0 * x) * (0 * y)$,

(4) $x * (x * (x * y)) = x * y$,

(5) $(x * z) * (y * z) \leq x * y$,

(6) $x * y = 0$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

In what follows, X shall mean a BCI-algebra unless otherwise specified.

Definition 2.3. ([6])

A non-empty subset I of X is called an BCI-*ideal* of X if it satisfies:

(I_1) $0 \in I$,

(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.4. ([13])

A BCI-algebra is said to be implicative if it satisfies: $(x * (x * y)) * (y * x) = y * (y * x)$.

Definition 2.5. ([11])
A nonempty subset I of X is called a sub-implicative ideal of X if it satisfies:
(I_1) $0 \in I$,
(I_3) $((x*(x*y))*(y*x))*z \in I$ and $z \in I$ imply $y*(y*x) \in I$ for all x, y, z $\in$ X.

Theorem 2.6. ([2])
Let I be an ideal of X . Then I is sub-implicative if and only if $((x*(x*y))*(y*x)) \in I$ implies $y*(y*x) \in I$.

Theorem 2.7. ([11])
Any sub-implicative ideal is an ideal, but the converse is not true.

Definition 2.8. ([17])
Let X be a non empty set. A fuzzy set μ of X is a function $\mu : X \to [0,1]$. Let μ be a fuzzy set of X. then for t∈ [0, 1] the t-level cut of μ is the set
$\mu_t = \{x \in X : \mu(x) \ge t\}$,and the complement of μ, denoted by μ^c, is the fuzzy set of X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

Definition 2.9. ([16])
A fuzzy set μ of a BCI-algebra X is called a fuzzy sub-algebra of X if $\mu(x*y) \ge \min\{\mu(x), \mu(y)\}$ for all x , y ∈X.

Definition 2.10. ([8])
A fuzzy set μ in a BCI-algebra X is said to be a fuzzy ideal in X if it satisfies
(F_1) $\mu(0) \ge \mu(x)$,
(F_2) $\mu(x) \ge \min\{\mu(x*y), \mu(y)\}$ for all x, y ∈ X.

Definition 2.11. ([9])
A fuzzy set μ of X is called a fuzzy sub-implicative ideals (briefly, FSI-ideals) of X if it satisfies:
(F_1) $\mu(0) \ge \mu(x)$ and
(F_3) $\mu(y*(y*x)) \ge \min\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$ for all x, y, z ∈ X.

Definition 2.12. ([5])
A fuzzy set μ of a BCI-algebra X is called an anti-fuzzy sub-algebra of X if :
$\mu(x*y) \le \max\{\mu(x), \mu(y)\}$ for all x , y ∈ X.

Definition 2.13. ([5])
A fuzzy set μ of a BCI-algebra X is called an anti-fuzzy ideal of X if it satisfies:
(AF_1) $\mu(0) \le \mu(x)$,
(AF_2) $\mu(x) \le \max\{\mu(x*y), \mu(y)\}$,for all x ,y∈ X.

Proposition 2.14. ([5])
Every anti-fuzzy ideal of a BCI-algebra X is an anti-fuzzy sub-algebra of X.

Definition 2.15. ([5])
Let μ be a fuzzy set of a BCI-algebra X. Then for $t \in [0,1]$ the lower t-level cut of μ is the set

$$\mu^t = \{x \in X \mid \mu(x) \le t\}.$$

Definition 2.16.([5])
Let μ be a fuzzy set of a BCI-algebra X. The fuzzification of μ^t, t ∈[0,1] is the fuzzy subset μ_{μ^t} of X defined by :

$$\mu_{\mu^t} = \begin{cases} \mu(x) & \text{if } x \in \mu^t \\ 0 & \text{other wise} \end{cases}.$$

3. Anti-fuzzy sub-implicative ideals

Definition 3.1.
A fuzzy set μ of a BCI-algebra X is called an anti-fuzzy sub-implicative ideal of X (briefly, AFSI-ideal) if it satisfies (AF_1) and (AF_3)
$\mu(y*(y*x)) \le \max\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$ for all x, y, z ∈ X.

Example 3.2. Let X = {0, 1, 2} be a BCI-algebra with Cayley table as follows:

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define $\mu : X \to [0,1]$ by $\mu(0) = \mu(1) = t_0$ and $\mu(2) = t_1$, where $t_0, t_1 \in [0,1]$ and $t_0 < t_1$. By routine calculations give that μ is an AFSI-ideal of X.

Proposition 3.3.
Every an anti-fuzzy sub-implicative ideal of a BCI-algebra X is order preserving.
Proof.
Let μ be AFSI-ideal of X and let x, y, z ∈X be such that $x \le z$, then $x*z = 0$ and by (AF_3) $\mu(y*(y*x)) \le \max\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$.--------- (W)
Let y = x in (W), and using (III), (2) , we get
$\mu(x) \le \max\{\mu(((x*(x*x))*(x*x))*z), \mu(z)\}$
$= \max\{\mu(x*z), \mu(z)\} = \max\{\mu(0), \mu(z)\}$
$= \mu(z)$.

Proposition 3.4.

Every anti-fuzzy sub-implicative ideal of BCI-algebra X is an anti -fuzzy ideal.

Proof.

Let μ be an anti-fuzzy sub-implicative ideal of a BCI-algebra X, for all x, y, z $\in$ X,
$\mu(y * (y * x)) \leq \max\{\mu(((x * (x * y)) * (y * x)) * z), \mu(z)\}$, put y = x, and using (III), (2) we get
$\mu(x) \leq \max\{\mu(((x * (x * x)) * (x * x)) * z), \mu(z)\}$
$= \max\{\mu(x * z), \mu(z)\}$, for all x, z $\in$ X.
Hence μ is an anti-fuzzy ideal of X.

The following example shows that the converse of proposition 3.4 may not be true.

Example 3.5.

Let X = {0,1,2,3} be a BCI-algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.2$ and $\mu(x) = 0.7$ for all $x \neq 0$. Then μ is an anti-fuzzy ideal of X, but it is not an anti fuzzy sub-implicative ideal of X because
$\mu(1 * (1 * 2)) > \max\{\mu(((2 * (2 * 1)) * (1 * 2)) * 0), \mu(0)\}$.

Proposition 3.6.

Let μ be an AFSI-ideal of BCI-algebra X. Then μ satisfies the inequality

$$\mu(y * (y * x)) \leq \mu((x * (x * y)) * (y * x)).$$

Proof. Clear.

We now give a condition for an anti-fuzzy ideal to be an anti-fuzzy sub-implicative ideal.

Theorem 3.7.

Every anti fuzzy-ideal μ of X satisfies the inequality $\mu(y * (y * x)) \leq \mu((x * (x * y)) * (y * x))$ for all x, y $\in$ X, is an anti-fuzzy sub-implicative ideal of X.

Proof.

Let μ be an anti-fuzzy ideal of X satisfying the inequality, $\mu(y * (y * x)) \leq \mu((x * (x * y)) * (y * x)) \leq \max\{\mu(((x * (x * y)) * (y * x)) * z), \mu(z)\}$ by (AF_2).which proves the condition (AF_3). This completes the proof .

Lemma 3.8.

Every AFSI-ideal of BCI-algebra is an anti-fuzzy sub-algebra of X.

Proof.

Let μ be an AFSI-ideal of BCI-algebra X, then $\mu(y * (y * x)) \leq \max\{\mu(((x * (x * y)) * (y * x)) * z), \mu(z)\}$, put y = x, we have
$\mu(x) \leq \max\{\mu(x * z), \mu(z)\}$,which imply that
$\mu(x * z) \leq \max\{\mu((x * z) * z), \mu(z)\}$.
But $(x * z) * z \leq x * z \leq x$, then
$\mu((x * z) * z) \leq \mu(x)$ [by proposition 3.3]. So $\mu(x * z) \leq \max\{\mu(x), \mu(z)\}$, then μ is an anti-fuzzy sub-algebra of X.

Lemma 3.9.

If X is implicative BCI-algebra, then every anti-fuzzy ideal of X is an AFSI-ideal of X.

Proof.

Let μ be an anti-fuzzy ideal of X, then
$\mu(x) \leq \max\{\mu(x * z), \mu(z)\}$ for all x, z $\in$ X.
So $\mu(y * (y * x)) \leq \max\{\mu((y * (y * x)) * z), \mu(z)\}$,
but X is implicative BCI-algebra, then
$(x * (x * y)) * (y * x) = y * (y * x)$, and hence
$\mu(y * (y * x)) \leq \max\{\mu(((x * (x * y)) * (y * x)) * z), \mu(z)\}$.Which shows that μ is AFSI-ideal of X.

By applying proposition(3.4) and lemma(3.8), we have the following Theorem:

Theorem 3.10.

If X is an implicative BCI-algebra, then a fuzzy set μ of X is an anti-fuzzy ideal of X if and only if it is an anti-fuzzy sub-implicative ideal of X.

Definition 3.11.

A fuzzy set μ in X is called an anti-fuzzy positive implicative if it satisfies:
(AF_1) $\mu(0) \leq \mu(x)$,
(AF_4) $\mu(x * z) \leq \max\{\mu(((x * z) * z) * (y * z)), \mu(z)\}$
for all x, y, z $\in$ X.

Analogous to (theorem 3.5 [11]), we have a similar result for an anti-fuzzy positive implicative ideal which can be proved in a similar manner, we state the result without proof.

Lemma 3.12.

Let μ be an anti-fuzzy ideal of X. Then the following are equivalent:
(i) μ is an anti-fuzzy positive implicative ideal of X ,
(ii) $\mu((x * y) * z) \leq \mu(((x * z) * z) * (y * z))$
for all x, y, z $\in$ X,
(iii) $\mu(x * y) \leq \mu(((x * y) * y) * (0 * y))$
for all x, y $\in$ X.

Theorem 3.13.

Every anti-fuzzy sub-implicative ideal of X is anti-fuzzy positive implicative ideal of X.

Proof.

Let μ be an AFSI-ideal of BCI-algebra X. Then μ is an anti-fuzzy ideal of X. for all x, y$\in$ X,

$\mu(x*y) = \mu(x*(x*(x*y)))$ [by Proposition 2.2.(4)]

$\leq \mu(((x*y)*((x*y)*x))*(x*(x*y)))$ [proposition3.6.]

$= \mu(((x*y)*(x*(x*y)))*((x*y)*x))$

$= \mu(((x*(x*(x*y))*y)*((x*x)*y))$

$= \mu(((x*y)*y)*(0*y))$,(by lemma 3.12), then μ is an anti fuzzy positive implicative ideal of X.

We can easily check that the anti-fuzzy set μ in Example 3.5 is an anti-fuzzy positive implicative ideal of X. Hence we know that the converse of Theorem 3.13 may not true.

Definition 3.14.

A fuzzy set μ in X is called anti fuzzy p-ideal of X if it satisfies:

(AF_1) $\mu(0) \leq \mu(x)$,

(AF_5) $\mu(x) \leq \max\{\mu((x*z)*(y*z)), \mu(y)\}$ for all x, y, z $\in$ X.

Remark(1)

Every anti-fuzzy p-ideal is anti fuzzy ideal, but the converse does not hold.

Remark(2)

Take z = x and y = 0 in (AF_5), then every anti-fuzzy p-ideal in X satisfies the inequality

$\mu(x) \leq \mu(0 * (0 * x))$ for all x $\in$X.

Theorem 3.15.

Every anti-fuzzy p-ideal of X is anti-fuzzy sub-implicative ideal of X.

Proof.

Let μ be an anti-fuzzy p-ideal of X. Then μ is an anti-fuzzy ideal of X, and .

$(0*(0*(y*(y*x))))*((x*(x*y))*(y*x))$

$=(0*((x*(x*y))*(y*x)))*(0*(y*(y*x)))$ [by(1)]

$=((0*(x*(x*y)))*(0*(y*x)))*((0*y)*(0*(y*x)))$ [by(3)]

$=(((0*x)*(0*(x*y)))*(0*(y*x)))*((0*y)*(0*(y*x)))$

$\leq ((0*x)*(0*(x*y)))*(0*y)$ [by(5)]

$= ((0*x)*(0*y))*(0*(x*y))$ [by(1)]

$= (0*(x*y))*(0*(x*y)) = 0$.

From remark(2) we have,

$\mu(y*(y*x)) \leq \mu(0 * (0 * (y*(y*x))))$. But $(0*(0*(y*(y*x)))) \leq ((x*(x*y))*(y*x))$. Since every anti-fuzzy ideal is order preserving, then $\mu(0*(0*(y*(y*x)))) \leq \mu((x*(x*y))*(y*x))$, hence $\mu(y*(y*x)) \leq \mu((x*(x*y))*(y*x))$. From theorem 3.7,we get μ is an anti-fuzzy sub-implicative ideal of X.

In the following example, we see that the converse of theorem 3.15 may not be true.

Example 3.16.

Consider a BCI-algebra X = {0,a,1,2,3} with Cayley table

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Define an anti-fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.2$, $\mu(a) = 0.5$ and $\mu(1) = \mu(2) = \mu(3) = 0.7$. Then μ is a anti-fuzzy ideal of X in which the inequality $\mu(y*(y*x)) \leq \mu((x*(x*y))*(y*x))$ holds for all x, y $\in$X. Using theorem 3.7, we see that μ is an anti-fuzzy sub-implicative ideal of X. μ is not anti-fuzzy p-ideal of X , since

$\mu(a) > \max\{\mu((a*1)*(0*1)), \mu(0)\}$.

Theorem 3.17.

For any AFSI-ideal μ of X ,the set

$X_\mu = \{x \in X \mid \mu(x)=\mu(0)\}$ is sub-implicative ideal of X.

Proof.

Clearly $0 \in X_\mu$. Let x, y, z $\in$X be such that

$((x*(x*y))*(y*x))*z \in X_\mu$ and $z \in X_\mu$.

By (AF_3), we have

$\mu(y*(y*x)) \leq \max\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$

$= \mu(0)$, which implies from (AF_1) that

$\mu(y*(y*x)) = \mu(0)$. Then $y*(y*x) \in X_\mu$,

therefore X_μ is a sub-implicative ideal of X.

Applying Theorems 3.15 and 3.17, we have the following corollary.

Corollary 3.18. If μ is an anti-fuzzy p-ideal of X, then the set $X_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$ is a sub-implicative ideal of X.

Theorem 3.19.

A fuzzy set μ of a BCI-algebra **X** is a fuzzy sub-implicative ideal of X if and only if its complement μ^c is an AFSI-ideal of X.

Proof.

Let μ be a fuzzy sub-implicative ideal of a BCI-algebra X, and let x, y, z $\in$ X, then

$\mu^c(0) = 1 - \mu(0) \le 1 - \mu(x) = \mu^c(x)$,and

$\mu^c(y*(y*x)) = 1 - \mu(y*(y*x))$
$\le 1 - \min[\mu(((x*(x*y))*(y*x))*z), \mu(z)]$
$= 1 - \min[1 - \mu^c(((x*(x*y))*(y*x))*z), 1 - \mu^c(z)]$
$= \max[\mu^c(((x*(x*y))*(y*x))*z), \mu^c(z)]$.

So, μ^c is an AFSI-ideal of X. Now let μ^c be an AFSI-ideal of X, and let x, y, z $\in$ X, then

$\mu(0) = 1 - \mu^c(0) \ge 1 - \mu^c(x) = \mu(x)$, and

$\mu(y*(y*x)) = 1 - \mu^c(y*(y*x))$
$\ge 1 - \max\{\mu^c(((x*(x*y))*(y*x))*z), \mu^c(z)\}$
$= 1 - \max\{1 - \mu(((x*(x*y))*(y*x))*z), 1 - \mu(z)\}$
$= \min\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$.

Thus, μ is a fuzzy sub-implicative ideal of X.

Theorem 3.20.

Let μ be a fuzzy set of BCI-algebra X. Then μ is an AFSI-ideal of X if and only if for each t $\in [0,1]$, t $\ge \mu(0)$, the lower t-level cut μ^t is a sub-implicative ideal of X .

Proof.

Let μ be an AFSI-ideal of X and let $t \in [0,1]$ with $\mu(0) \le t$. By (AF_1), we have $\mu(0) \le \mu(x)$ for all $x \in X$, but $\mu(x) \le t$ for all $x \in \mu^t$ and so $0 \in \mu^t$. Let x, y, z $\in$ X be such that $((x*(x*y))*(y*x))*z \in \mu^t$ and $z \in \mu^t$, then $\mu(((x*(x*y))*(y*x))*z) \le t$ and $\mu(z) \le t$. Since μ is an AFSI-ideal, it follow that

$\mu(y*(y*x)) \le \max\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$ $\le t$, and hence $y*(y*x) \in \mu^t$ .Therefore μ^t is sub-implicative ideal of X.

Conversely, let μ^t be a sub-implicative ideal of X. We only need to show that (AF_1) ,(AF_3) are true.

If (AF_1) is false, then there exist $x_0 \in X$ such that $\mu(0) > \mu(x_0)$. If wetake $t_0 = \frac{1}{2}\{\mu(0) + \mu(x_0)\}$, then $\mu(0) > t_0$ and $0 \leqslant \mu(x_0) < t_0 \leqslant 1$.

Hence $x_0 \in \mu^{t_0}$ and $\mu^{t_0} \ne \phi$. But μ^{t_0} is sub-implicative ideal of X, we have $0 \in \mu^{t_0}$ and so $\mu(0) \le t_0$, contradiction. Hence $\mu(0) \le \mu(x)$ for all $x \in X$. Now, assume (AF_3) is not true, then there exist $x_0, y_0, z_0 \in X$ such that

$\mu(y_0*(y_0*x_0)) > \max\{\mu(((x_0*(x_0*y_0))*(y_0*x_0))*z_0), \mu(z_0)\}$. Putting

$s_0 = \frac{1}{2}\{\mu(y_0*(y_0*x_0)) + \max\{\mu(((x_0*(x_0*y_0))*(y_0*x_0))*z_0), \mu(z_0)\}\}$, then $s_0 < \mu(y_0*(y_0*x_0))$ and

$0 \le \max\{\mu(((x_0*(x_0*y_0))*(y_0*x_0))*z_0), \mu(z_0)\} < s_0 \le 1$. Thus we have

$\max\{\mu(((x_0*(x_0*y_0))*(y_0*x_0))*z_0) < s_0, \mu(z_0) < s_0$,

but μ^{s_0} is an sub-implicative ideal of X, thus $y_0*(y_0*x_0) \in \mu^{s_0}$ or $\mu(y_0*(y_0*x_0)) \le s_0$ .This a contradiction, ending the proof.

Theorem 3.21.

If μ is an AFSI-ideal of a BCI-algebra X. then μ_{μ^t} is also an AFSI-ideal of X ,where $t \in [0,1]$ and $t \ge \mu(0)$.

Proof.

From the theorem 3.20, it is sufficient to show that $(\mu_{\mu^t})^{\delta}$ is a sub-implicative ideal of X , where $\delta \in [0,1]$ and $\delta \ge \mu_{\mu^t}(0)$.

Clearly, $0 \in (\mu_{\mu^t})^{\delta}$. Let x, y, z $\in X$ be such that $((x*(x*y))*(y*x))*z \in (\mu_{\mu^t})^{\delta}$ and $z \in (\mu_{\mu^t})^{\delta}$.

Thus $\mu_{\mu^t}(((x*(x*y))*(y*x))*z) \le \delta$ and $\mu_{\mu^t}(z) \le \delta$ .We claim that $y*(y*x) \in (\mu_{\mu^t})^{\delta}$ or $\mu_{\mu^t}(y*(y*x)) \le \delta$ .If

$((x*(x*y))*(y*x))*z \in \mu^t$ and $z \in \mu^t$,then $y*(y*x) \in \mu^t$, since μ^t is a sub-implicative ideal of X. we have

$\mu_{\mu^t}(y*(y*x)) = \mu(y*(y*x))$
$\le \max\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$
$= \max\{\mu_{\mu^t}(((x*(x*y))*(y*x))*z), \mu_{\mu^t}(z)\} \le \delta$

and so $y*(y*x) \in (\mu_{\mu^t})^{\delta}$.

If $((x*(x*y))*(y*x))*z \notin \mu^t$ or $z \notin \mu^t$, then $\mu_{\mu^t}(((x*(x*y))*(y*x))*z) = 0$ or $\mu_{\mu^t}(z) = 0$, then clearly $\mu_{\mu^t}(y*(y*x)) \le \delta$ and so $y*(y*x) \in (\mu_{\mu^t})^{\delta}$. There for $(\mu_{\mu^t})^{\delta}$ is a sub-implicative ideal of X.

Definition 3.22.

A fuzzy set μ of a BCI-algebra X is called an anti-fuzzy sub-commutative ideal of X (briefly, AFSC-ideal) if it satisfies (AF_1) and (AF_6)

$\mu(x*(x*y)) \le \max\{\mu((y*(y*(x*(x*y))))*\mu(z)\}$

for all x, y, z $\in$ X.

Theorem 3.23.

Every anti-fuzzy sub-implicative ideal of X is anti-fuzzy sub-commutative ideal of X, but the converse is not true.

Proof.

Let μ be an AFSI-ideal of X. Then it satisfies (AF_1) and by (AF_3) we have $\mu(x*(x*y)) \le \max\{\mu(((y*(y*x))*(x*y))*z), \mu(z)\}$ for all $x, y, z \in X$. But by using (1) and (4) we have $[(y*(y*x))*(x*y)] * [y*(y*(x*(x*y))) =$ $[(y*(y*(y*(x*(x*y)))))*(y*x)]*(x*y) =$ $[(y*(x*(x*y)))*(y*x)]*(x*y) =$ $[(y*(y*x))*(x*(x*y))]*(x*y) \le$ $(x*(x*(x*y)))*(x*y) = (x*(x*y))*(x*(x*y)) =$ 0, we have $(y*(y*x))*(x*y) \le y*(y*(x*(x*y)))$, which imply that $((y*(y*x))*(x*y))*z \le$ $(y*(y*(x*(x*y))))*z$, (by proposition 3.3) we get $\mu(((y*(y*x))*(x*y))*z) \le \mu((y*(y*(x*(x*y))))*z)$. So $\mu(x*(x*y)) \le \max\{\mu((y*(y*(x*(x*y))))*z), \mu(z)\}$, hence μ an AFSC-ideal of X. The last part of the theorem is shown by the following example:

Example 3.24.
Let X = {0,1,2,3} be a BCI-algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let μ be a fuzzy set in X defined by $\mu(0) = \mu(3) = 0.2$ and $\mu(1) = \mu(2) = 0.8$. It is easy to verify that μ is an AFSC- ideal of X, but it is not an AFSI-ideal of X since $\mu((1*(1*2))*(2*1)) = \mu(0) = 0.2 < 0.8 = \mu(1) = \mu(2*(2*1))$. The proof is complete.

4.Homomorphism of AFSI-ideal of BCI-algebra

Definition 4.1.

Let f be a mapping of BCI-algebra X into BCI-algebra Y and $A \subseteq X$, $B \subseteq Y$. The image of A in Y is $f(A) = \{f(a) \mid a \in A\}$ and the inverse image of B is $f^{-1}(B) = \{g \in X \mid f(g) \in B\}$.

Definition 4.2.

Let $(X, *, 0)$ and $(Y, *^{\backslash}, 0^{\backslash})$ be a BCI-algebras. A mapping $f: X \to Y$ is said to be a homomorphism if $f(x*y) = f(x) *^{\backslash} f(y)$ for all $x, y \in X$.

Theorem 4.3.
Let f be a homomorphism of BCI-algebra X into a BCI-algebra Y, then:
(i) If 0 is the identity in X, then $f(0)$ is the identity in Y.
(ii) If A is sub-implicative ideal of X, then $f(A)$ is sub-implicative ideal of Y.
(iii) If B is sub-implicative ideal of Y, then $f^{-1}(B)$ is sub-implicative ideal of Y.
(iv) If X is implicative BCI-algebra, then $\ker f$ is sub-implicative ideal of X.

Proof.
(i) By using Definition 2.1 and Definition 4.2, we have $f(0) = f(0*0) = f(0) *^{\backslash} f(0) = 0^{\backslash}$.
(ii) Let A be an sub-implicative ideal of X. Clearly $0^{\backslash} \in f(A)$ .If $((f(x) *^{\backslash} (f(x) *^{\backslash} f(y))) *^{\backslash} (f(y) *^{\backslash} f(x))) *^{\backslash} f(z) \in f(A)$ and $f(z) \in f(A)$, then $f(((x*(x*y))*(y*x))*z) \in f(A)$,since f is a homomorphism, we have $((x*(x*y))*(y*x))*z \in A$ and $z \in A$. Since A is sub-implicative ideal, then $y*(y*x) \in A$ and hence $f(y*(y*x)) = f(y) *^{\backslash} (f(y) *^{\backslash} f(x)) \in f(A)$. We have $f(A)$ is sub-implicative ideal of Y.
(iii) Let B be an sub-implicative ideal of $f(X)$, since $f(0) = 0^{\backslash}$, $0 \in f^{-1}(B)$.
Let $((x*(x*y))*(y*x))*z \in f^{-1}(B)$, $z \in f^{-1}(B)$ for all $x, y, z \in X$, then $f(((x*(x*y))*(y*x))*z) \in B$, $f(z) \in B$. But f is homomorphism, then $((f(x) *^{\backslash} (f(x) *^{\backslash} f(y))) *^{\backslash} (f(y) *^{\backslash} f(x))) *^{\backslash} f(z) \in B$ and $f(z) \in B$, since B is sub- implicative ideal, we have $f(y) *^{\backslash} (f(y) *^{\backslash} f(x)) = f(y*(y*x)) \in B$, and hence $y*(y*x) \in f^{-1}(B)$, then $f^{-1}(B)$ is sub-implicative ideal.
(iv) Let $x, y, z \in X$ be such that $((x*(x*y))*(y*x))*z \in \ker f$, $z \in \ker f$, then $f(((x*(x*y))*(y*x))*z) = 0^{\backslash}$, $f(z) = 0^{\backslash}$, since f is homomorphism we have
$((f(x) *^{\backslash} (f(x) *^{\backslash} f(y))) *^{\backslash} (f(y) *^{\backslash} f(x))) *^{\backslash} f(z) = 0^{\backslash}$
$((f(x) *^{\backslash} (f(x) *^{\backslash} f(y))) *^{\backslash} (f(y) *^{\backslash} f(x))) *^{\backslash} 0^{\backslash} =$
$((f(x) *^{\backslash} (f(x) *^{\backslash} f(y))) *^{\backslash} (f(y) *^{\backslash} f(x))) =$
$f((x*(x*y))*(y*x)) = 0^{\backslash}$,
but X is implicative BCI-algebra, then $f(y*(y*x)) = 0^{\backslash}$ i.e. $y*(y*x) \in \ker f$.
Then $\ker f$ is sub-implicative ideal of X.

Definition 4.4.

Let $f : X \rightarrow Y$ be a homomorphism of BCI-algebras and β be a fuzzy set of Y, then β^f is called the pre-image of β under f and its denoted by $\beta^f(x) = \beta(f(x))$, for all $x \in X$.

Theorem 4.5.
Let $f : X \rightarrow Y$ be a homomorphism of BCI-algebras. If β is an AFSI-ideal of Y, then β^f is an AFSI-ideal of X.

Proof.
Since β is an AFSI-ideal of Y, then $\beta(0^\backslash) \le \beta(f(x))$ for every $x \in X$ and so $\beta^f(0) = \beta(f(0)) = \beta(0^\backslash) \le \beta(f(x)) = \beta^f(x)$.
For any $x, y, z \in X$, we have
$\beta^f(y*(y*x)) = \beta(f(y*(y*x))$
$= \beta(f(y)*^\backslash(f(y)*^\backslash f(x)))$
$\le \max[\beta(((f(x)*^\backslash(f(x)*^\backslash f(y)))*^\backslash(f(y)*^\backslash f(x)))*^\backslash f(z)), \beta(f(z))]$
$= \max[\beta(f(((x*(x*y))*(y*x))*z), \beta(f(z))]$
$= \max[\beta^f((((x*(x*y))*(y*x))*z), \beta^f((z))]$.
Then β^f is AFSI-ideal of X.

Theorem 4.6.
Let $f : X \rightarrow Y$ be an epimorphism of BCI-algebras. If β^f is an anti-fuzzy sub-implicative ideal of X, then β is an AFSI-ideal of Y.

Proof.
Let β^f be an AFSI-ideal of X and $y \in Y$, there exist $x \in X$ such that $f(x) = y$. Then $\beta(y) = \beta(f(x)) = \beta^f(x) \ge \beta^f(0) = \beta(f(0)) = \beta(0^\backslash)$. Let $x^\backslash, y^\backslash, z^\backslash \in Y$, then there exist $x, y, z \in X$ such that $f(x) = x^\backslash$, $f(y) = y^\backslash$ and $f(z) = z^\backslash$. It follows that $\beta(y^\backslash *^\backslash (y^\backslash *^\backslash x^\backslash)) = \beta(f(y) *^\backslash (f(y) *^\backslash f(x))) = \beta(f(y*(y*x)) = \beta^f(y*(y*x)) \le$
$\max\{\beta^f(((x*(x*y)) * (y*x))*z), \beta^f(z)\}$
$= \max\{\beta(f(((x*(x*y))*(y*x))*z), \beta(f(z))\}$
$= \max\{\beta(((f(x) *^\backslash (f(x) *^\backslash f(y)))*^\backslash(f(y)*^\backslash f(x)))*^\backslash f(z)), \beta(f(z))\} =$
$\max\{\beta(((x^\backslash *^\backslash (x^\backslash *^\backslash y^\backslash))*^\backslash(y^\backslash *^\backslash x^\backslash))*^\backslash z^\backslash), \beta(z^\backslash)\}$
and hence β is an anti-fuzzy sub-implicative ideal of Y.

5. Cartesian product of AFSI-ideals

Definition 5.1. ([1])
A fuzzy relation on any set X is a fuzzy subset $\mu : X \times X \rightarrow [0,1]$.

Definition 5.2.
If μ is a fuzzy relation on a set X and β is a fuzzy subset of X, then μ is an anti-fuzzy relation on β if $\mu(x, y) \ge \max\{\beta(x), \beta(y)\}$ for all $x, y \in X$.

Definition 5.3.
Let μ and λ be anti-fuzzy subsets of a set X. The Cartesian product $\mu \times \lambda : X \times X \rightarrow [0,1]$ is defined by $(\mu \times \lambda)(x, y) = \max\{\mu(x), \lambda(y)\}$ for all $x, y \in X$.

Iemma 5.4. ([1])
Let μ and λ be fuzzy subsets of a set X. Then ,
(i) $\mu \times \lambda$ is a fuzzy relation on X,
(ii) $(\mu \times \lambda)_t = \mu_t \times \lambda_t$ for all $t \in [0,1]$.

Definition 5.5.
If β is a fuzzy set of a set X, the strongest anti fuzzy relation on X that is an anti-fuzzy relation on β is μ_β given by $\mu_\beta(x, y) = \max\{\beta(x), \beta(y)\}$ for all $x, y \in X$.

Proposition 5.6.
For a given fuzzy set β of a BCI-algebra X, let μ_β be the strongest anti-fuzzy relation on X. If μ_β is an anti-fuzzy sub-implicative ideal of $X \times X$, then $\beta(x) \ge \beta(0)$ for all $x \in X$.

Proof.
$\mu_\beta(x, x) = \max\{\beta(x), \beta(x)\} \ge \mu_\beta(0,0) = \max\{\beta(0), \beta(0)\}$ where $(0, 0) \in X \times X$, then $\beta(x) \ge \beta(0)$ for all $x \in X$.

Remark 5.7.
Let X and Y be BCI-algebras, we define $*$ on $X \times Y$ by, for every $(x, y), (u, v) \in X \times Y$, $(x, y) * (u, v) = (x * u, y * v)$. Then clearly $(X \times Y; *, (0, 0))$ is a BCI-algebra.

Theorem 5.8.
Let μ and β be AFSI-ideals of BCI-algebra X. Then $\mu \times \beta$ is an anti-fuzzy sub-implicative ideal of $X \times X$.

Proof.
Let μ and β be AFSI-ideals of BCI-algebra X, for every $(x, y) \in X \times X$, we have
$(\mu \times \beta)(0,0) = \max\{\mu(0), \beta(0)\}$
$\le \max\{\mu(x), \beta(y)\} = (\mu \times \beta)(x, y)$.
Now we let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X \times X$, we

have $(\mu \times \beta)((y_1, y_2) * ((y_1, y_2) * (x_1, x_2))) = (\mu \times \beta)(y_1 * (y_1 * x_1), y_2 * (y_2 * x_2)) =$
$\max\{\mu(y_1 * (y_1 * x_1)), \beta(y_2 * (y_2 * x_2))\} \le$
$\max\{\max\{\mu(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1), \mu(z_1)\},$
$\max\{\beta(((x_2 * (x_2 * y_2)) * (y_2 * x_2)) * z_2), \beta(z_2)\}\} =$
$\max\{\max\{\mu(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1),$
$\beta(((x * (x_2 * y_2 * (y_2 * x_2)) * z_2)\}, \max\{\mu(z_1), \beta(z_2)\}\} =$
$\max\{(\mu \times \beta)(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1, ((x_2 * (x_2 * y_2)) * (y_2 * x_2)) * z_2), (\mu \times \beta)(z_1, z_2)\} =$
$\max\{(\mu \times \beta)((((x_1,x_2) * ((x_1,x_2) * (y_1,y_2))) * ((y_1,y_2) * (x_1,x_2))) * (z_1,z_2)), (\mu \times \beta)(z_1, z_2)\}.$

Analogous to theorem 3.2[15], we have a similar result for AFSI-ideals, which can beproved in a similar manner, we state the result without proof.

Theorem 5.9.

Let μ and β be a fuzzy sets of a BCI-algebra X such that $\mu \times \beta$ is an AFSI-ideal of $X \times X$. Then,

(i) Either $\mu(x) \ge \mu(0)$ or $\beta(x) \ge \beta(0)$ for all $x \in X$,

(ii) If $\mu(x) \ge \mu(0)$ for all $x \in X$, then either $\mu(x) \ge \beta(0)$ or $\beta(x) \ge \mu(0)$,

(iii) If $\beta(x) \ge \beta(0)$ for all $x \in X$, then either $\mu(x) \ge \mu(0)$ or $\beta(x) \ge \mu(0)$,

(iv) Either μ or β is an AFSI-ideal of X.

Theorem 5.10.

Let β be a fuzzy set of a BCI-algebra X and let μ_β be the strongest anti-fuzzy relation on X. Then β is an AFSI-ideal of X if and only if μ_β is an anti-fuzzy sub-implicative ideal of $X \times X$.

Proof.

Assume that β is an AFSI-ideal of X. We note from (AF_1) that $\mu_\beta(0,0) = \max\{\beta(0), \beta(0)\} \le \max\{\beta(x), \beta(y)\} = \mu_\beta(x, y)$ for all $(x, y) \in X \times X$.

For all $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X \times X$, we get

$\mu_\beta((y_1, y_2) * ((y_1, y_2) * (x_1, x_2)))$
$= \mu_\beta(y_1 * (y_1 * x_1), y_2 * (y_2 * x_2))$
$= \max\{\beta(y_1 * (y_1 * x_1)), \beta(y_2 * (y_2 * x_2))\}$
$\le \max\{\max\{\beta(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1), \beta(z_1)\},$
$\max\{\beta(((x_2 * (x_2 * y_2)) * (y_2 * x_2)) * z_2), \beta(z_2)\}\}$
$= \max\{\max\{\beta(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1),$
$\beta(((x_2 * (x_2 * y_2)) * (y_2 * x_2) * z_2)\}, \max\{\beta(z_1), \beta(z_2)\}\}$
$= \max\{\mu_\beta(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1, ((x_2 * (x_2 * y_2)) * (y_2 * x_2)) * z_2), \mu_\beta(z_1, z_2)\}$
$= \max\{\mu_\beta((((x_1,x_2) * ((x_1,x_2) * (y_1,y_2))) * ((y_1,y_2) * (x_1,x_2))) * (z_1,z_2)), \mu_\beta(z_1, z_2)\}.$

Hence, μ_β is an anti-fuzzy sub-implicative ideal of $X \times X$. Conversely, suppose that μ_β is an AFSI-ideal of $X \times X$. Then for all $(x, y) \in X \times X$,
$\max\{\beta(0), \beta(0)\} = \mu_\beta(0, 0) \le \mu_\beta(x, y) = \max\{\beta(x), \beta(y)\}$ follows that $\beta(0) \le \beta(x)$ for all $x \in X$, which proves (AF_1).

Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. Then,
$\max\{\beta(y_1 * (y_1 * x_1)), \beta(y_2 * (y_2 * x_2))\} = \mu_\beta(y_1 * (y_1 * x_1), y_2 * (y_2 * x_2)) =$
$\mu_\beta((y_1, y_2) * ((y_1, y_2) * (x_1, x_2))) \le$
$\max\{\mu_\beta((((x_1,x_2) * ((x_1,x_2) * (y_1,y_2))) * ((y_1,y_2) * (x_1,x_2))) * (z_1,z_2)), \mu_\beta(z_1, z_2)\} =$
$\max\{\mu_\beta(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1, ((x_2 * (x_2 * y_2)) * (y_2 * x_2)) * z_2), \mu_\beta(z_1, z_2)\} =$
$\max\{\max\{\beta(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1),$
$\beta(((x_2 * (x_2 * y_2)) * (y_2 * x_2) * z_2)\}, \max\{\beta(z_1), \beta(z_2)\}\} =$
$\max\{\max\{\beta(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1), \beta(z_1)\},$
$\max\{\beta(((x_2 * (x_2 * y_2)) * (y_2 * x_2)) * z_2), \beta(z_2)\}\}.$

Take $x_2 = y_2 = z_2 = 0$, then $\beta(y_1 * (y_1 * x_1)) \le \max\{\beta(((x_1 * (x_1 * y_1)) * (y_1 * x_1)) * z_1), \beta(z_1)\}$. Then β is an anti-fuzzy sub-implicative ideal of X.

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