Some Properties of Doubt Fuzzy Sub-Commutative Ideals of BCI-Algebras

Samy M. Mostafa, Ragab A.K. Omar and Ahmed I. Marie

Department of Mathematic, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt
samymostafa@yahoo.com; ahmedibrahim500@yahoo.com

Abstract: In this paper, we introduce the notion of doubt fuzzy sub-commutative ideal of BCI-algebras, and study some of their properties. We show that a fuzzy subset of BCI-algebra is a fuzzy sub - commutative ideal if and only if the complement of this fuzzy subset is a doubt fuzzy sub-commutative ideal, and any doubt fuzzy ideal of commutative BCI-algebra is doubt fuzzy sub-commutative ideal. We investigate how to deal with the homomorphic image (pre-image) of doubt fuzzy sub-commutative ideal of BCI-algebra. Moreover, we introduce the notion of Cartesian product of doubt fuzzy sub-commutative ideals and then we study some related properties.

[Samy M. Mostafa, Ragab A.K. Omar and Ahmed I. Marie **Some Properties of Doubt Fuzzy Sub-Commutative Ideals of BCI-Algebras**] Journal of American Science 2011;7(11):283-291]. (ISSN: 1545-1003). http://www.americanscience.org.

Keywords: ideal of BCI-algebras - doubt fuzzy sub-commutative ideal - homomorphic image (pre-image) of sub-commutative ideal - Cartesian product of doubt fuzzy sub-commutative ideal.

2000 Mathematics Subject Classification: 03G25, 06F35, 03E72.

1. Introduction

The concept of a fuzzy set was introduced by Zadeh [14] and was used afterwards by many other outhers in various branches of mathematics. Isaki and Tanaka [4] introduced two classes of abstract algebras BCI-algebras and BCK-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI- algebra. Xi [13] applied the concept of fuzzy set to BCI-algebras and gave some properties of it. After that several researchers investigated further properties of fuzzy BCI-algebras and fuzzy ideal [see {[1] , [2] , [9] , [11] , [12]}]. Jun [6] gave some properties of a fuzzy commutative ideals in BCK-algebra .Liu and Meng [8] introduced the notion of sub-implicative ideal and sub-commutative ideal in BCI - algebra and investigated the properties of this ideals. Jun [7] defined a doubt fuzzy subalgebra, doubt fuzzy ideal, doubt fuzzy implicative ideal, and doubt fuzzy prime ideal in BCI- algebras, and got some results about it. Modifying this idea, in this paper, we introduce the concept of doubt fuzzy sub-commutative ideal of BCI-algebra and investigate some related properties. We show that in commutative BCI-algebra a fuzzy subset is an doubt fuzzy ideal if and only if it is doubt fuzzy sub-commutative ideal, and a fuzzy subset of a BCI-algebra is a fuzzy sub-commutative ideal if and only if the complement of this fuzzy subset is an doubt fuzzy sub-commutative ideal. Moreover, we discuss the homomorphic pre-image (image) of doubt fuzzy sub-commutative ideal. Finally, we introduce the notion of Cartesian product of doubt fuzzy sub-commutative ideal and then we characterize doubt fuzzy sub-commutative ideal by it.

2. Preliminaries

Definition 2.1. ([4])

An algebra $(X; *, 0)$ of type $(2,0)$ is called a BCI-algebra if it satisfies the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y, z \in X$.

We can define a partially ordered relation $\leq$ on X as follows:

$$x \leq y \text{ if and only if } x * y = 0.$$

Proposition 2.2. ([4])

A BCI-algebra X satisfies the following properties:

(1) $(x * y) * z = (x * z) * y$,

(2) $x * (x * (x * y)) = x * y$,

(3) $((x * z) * (y * z)) * (x * y) = 0$,

(4) $x * 0 = x$,

(5) $0 * (x * y) = (0 * x) * (0 * y)$,

(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

In what follows, X shall mean a BCI-algebra unless otherwise specified.

Definition 2.3. ([10])

A BCI-algebra X is said to be commutative if it satisfies:

$x \le y$ implies $x = y * (y * x)$ for all $x, y \in X$.

Definition 2.4. ([4])
A non-empty subset I of X is called an BCI-*ideal* of X if it satisfies:
(I_1) $0 \in I$,
(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

Theorem 2.5. ([10, theorem 3])
A BCI-algebra X is commutative if and only if
$x * (x * y) = y * (y * (x * (x * y)))$ for all $x, y \in X$.

Definition 2.6. ([8])
A nonempty subset I of X is called a sub-commutative ideal of X if it satisfies:
(I_1) $0 \in I$,
(I_3) $(y * (y * (x * (x * y)))) * z \in I$ and $z \in I$ imply $x * (x * y) \in I$ for all $x, y, z \in X$.

Theorem 2.7. ([8])
Let I be an ideal of X. Then I is sub-commutative if and only if
$y * (y * (x * (x * y))) \in I$ implies $x * (x * y) \in I$, for all $x, y \in X$.

Theorem 2.8. ([8])
Any sub-commutative ideal is an ideal, but the converse is not true.

Definition 2.9. ([14])
Let X be a non empty set. A fuzzy set μ of X is a function $\mu : X \to [0,1]$. Let μ be a fuzzy set of X. Then for $t \in [0, 1]$ the t-level cut of μ is the set $\mu_t = \{ x \in X : \mu(x) \ge t \}$, and the complement of μ, denoted by μ^c, is the fuzzy subset of X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

Definition 2.10. ([13])
A fuzzy set μ of a BCI-algebra X is called a fuzzy sub-algebra of X if $\mu(x * y) \ge \mu(x) \wedge \mu(y)$ for all $x, y \in X$.

Definition 2.11. ([13])
A fuzzy set μ of a BCI-algebra X is said to be a fuzzy ideal of X if it satisfies
(F_1) $\mu(0) \ge \mu(x)$,
(F_2) $\mu(x) \ge \mu(x * y) \wedge \mu(y)$ for all $x, y \in X$.

Lemma 2.12. ([11])
A fuzzy set μ of a BCI-algebra X satisfying (F_1) is a fuzzy ideal if and only if for all $x, y, z \in X$, $(x * y) * z = 0$ implies $\mu(x) \ge \mu(x) \wedge \mu(z)$.

Definition 2.13. ([9])
A fuzzy set μ of X is called a fuzzy sub-commutative ideal (briefly, FSC-ideals) of X if it satisfies
(F_1) $\mu(0) \ge \mu(x)$, and (F_3)
$\mu(x * (x * y)) \le \mu((y * (y * (x * (x * y)))) * z) \wedge \mu(z)$
for all $x, y, z \in X$.

Theorem 2.14.
Every fuzzy ideal of a commutative BCI-algebra X is a fuzzy sub-commutative ideal of X.
Proof.
Let μ be a fuzzy ideal of a commutative BCI-algebra X. Then (F_1) hold, as X is a commutative BCI-algebra, we have
$x * (x * y) = y * (y * (x * (x * y)))$
[Th. 2.4], then
$[(x * (x * y)) * ((y * (y * (x * (x * y)))) * z)] * z$
$= [(x * (x * y)) * ((x * (x * y)) * z)] * z$
$= [(x * (x * y)) * z] * [(x * (x * y)) * z] = 0$
by (1) and (III). By (lemma 2.12), we get
$\mu(x * (x * y)) \ge \mu((y * (y * (x * (x * y)))) * z) \wedge \mu(z)$. This shows that μ is a fuzzy sub-commutative ideal of X.

Definition 2.15. ([7])
A fuzzy set μ of a BCI-algebra X is called a doubt fuzzy sub- algebra of X if
$\mu(x * y) \le \mu(x) \vee \mu(y)$ for all $x, y \in X$.

Definition 2.16. ([7])
A fuzzy set μ of a BCI-algebra X is called a doubt fuzzy ideal of X if it satisfies:
(DF_1) $\mu(0) \le \mu(x)$,
(DF_2) $\mu(x) \le \mu(x * y) \vee \mu(y)$ for all $x, y \in X$.

Proposition 2.17. ([7])
Every doubt fuzzy ideal of a BCI-algebra X is a doubt fuzzy sub-algebra of X.

Proposition 2.18.
Every doubt fuzzy ideal of a BCI-algebra X is order preserving.

Proof. Let μ be a doubt fuzzy ideal of X and $x \le y$, then $x * y = 0$, and for all $x, y \in X$. we have

$\mu(x) \le \mu(x * y) \vee \mu(y) = \mu(0) \vee \mu(y) = \mu(y)$.

Definition 2.19. ([2])

Let μ be a fuzzy subset of a BCI-algebra X. Then for t ∈ [0,1] the lower t-level cut of μ is the set $\mu^t = \{x \in X \mid \mu(x) \le t\}$.

Definition 2.20. ([2])

Let μ be a fuzzy subset of a BCI-algebra X. The fuzzification of μ^t, t ∈[0,1] is the fuzzy subset μ_{μ^t} of X defined by

$$\mu_{\mu^t} = \begin{cases} \mu(x) & if\ x \in \mu^t \\ 0 & other\ wise \end{cases}.$$

3. Doubt fuzzy sub-commutative ideals

Definition 3.1.

A fuzzy set μ of a BCI-algebra X is called a doubt fuzzy sub-commutative ideal of X (briefly, DFSC-ideal) if it satisfies (DF_1) and (DF_3)

$\mu(x * (x * y)) \le \mu((y * (y * (x * (x * y)))) * z) \vee \mu(z)$

for all x, y, z ∈ X.

Example 3.2.

Let X = {0, 1, 2, 3} be a BCI-algebra with Cayley table as follows :

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Define $\mu : X \to [0,1]$ by $\mu(0) = \mu(3) = 0.3$ and $\mu(2) = \mu(1) = 0{,}7$. It is to check that μ is a doubt fuzzy sub-commutative ideal of X.

Proposition 3.3.

Every doubt fuzzy sub-commutative ideal of a BCI-algebra X is order preserving.

Proof.

Let μ be doubt fuzzy sub-commutative ideal of X and let x, y, z ∈ X be such that $x \le z$, then $x * z = 0$ and by (DF_3)

$\mu(x * (x * y)) \le \mu((y * (y * (x * (x * y)))) * z) \vee \mu(z)$.

Let y = x, then we have

$\mu(x) \le \mu((x * (x * (x * (x * x)))) * z) \vee \mu(z) =$

$\mu(x * z) \vee \mu(z) = \mu(0) \vee \mu(z) =$

$\mu(z)$.[by(III),(2)]

Proposition 3.4.

Every doubt fuzzy sub-commutative ideal of BCI-algebra X is a doubt fuzzy ideal.

Proof.

Let μ be a doubt fuzzy sub- commutative ideal of a BCI-algebra X for all x, y, z ∈ X , then $\mu(x * (x * y)) \le \mu((y * (y * (x * (x * y)))) * z) \vee \mu(z)$, put y = x , we get

$\mu(x) \le \mu((x * (x * (x * (x * x)))) * z) \vee \mu(z)$

$= \mu(x * z) \vee \mu(z)$ [by (III),(2)] ,

for all x, z ∈ X. Hence μ is doubt fuzzy ideal of X.

Lemma 3.5.

Every doubt fuzzy sub-commutative ideal of BCI-algebra is a doubt fuzzy sub algebra of X.

Proof.

Let μ be a doubt fuzzy sub-commutative ideal of BCI-algebra X, then

$\mu(x * (x * y)) \le \mu((y * (y * (x * (x * y)))) * z) \vee \mu(z)$, put y = x, we have

$\mu(x) \le \mu((x * (x * (x * (x * x)))) * z) \vee \mu(z)$

$= \mu(x * z) \vee \mu(z)$, [by (III),(2)]

for all x, z ∈ X, which imply that

$\mu(x * z) \le \mu((x * z) * z) \vee \mu(z)$, but

$(x * z) * z \le x * z \le x$, then

$\mu((x * z) * z) \le \mu(x)$ [by proposition 3.3].

So $\mu(x * z) \le \mu(x) \vee \mu(z)$, then μ is doubt fuzzy sub-algebra of X.

The following example shows that the converse of proposition 3.4 and lemma 3.5may not be true.

Example 3.6.

Let X = {0,1,2,3} be a BCI-algebra with Cayley table as follows:

*	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Define a fuzzy subset $\mu : X \to [0,1]$ by $\mu(0) = 0.5$ and $\mu(x) = 1$ for all $x \ne 0$. Then μ is doubt fuzzy ideal(sub algebra) of X, but it is not doubt fuzzy sub-commutative ideal of X, because $\mu(1 * (1 * 2)) = \mu(1) = 1 >$

$$\mu((2 * (2 * (1 * (1 * 2)))) * 0) \vee \mu(0) = \frac{1}{2}.$$

Analogous to (theorem 4.3 [9]), we have a similar result for a doubt fuzzy sub-commutative ideal which can be proved in a similar manner, we state the result withoutproof.

Theorem 3.7.

Let μ be a doubt fuzzy ideal of X. Then the following are equivalent:

(i) μ is doubt fuzzy sub-commutative ideal of X ,

(ii) μ (x * (x * y)) ≤ μ (y * (y *(x *(x * y)))) for all x, y ∈ X

(iii) μ (x * (x * y)) = μ (y * (y *(x *(x * y)))) for all x, y∈ X

(iv) If x ≤ y, then μ (x) = μ (y * (y * x)) for all x, y∈ X

(v) If x ≤ y, then μ (x) ≤ μ (y * (y * x)) for all x, y∈ X.

Proposition 3.8.

If X is commutative BCI-algebra, then every doubt fuzzy ideal of X is a doubt fuzzy sub-commutative ideal of X.

Proof.

Let μ be a doubt fuzzy ideal of X , then

μ (x) ≤ μ (x∗z) ∨ μ (z) for all x, z ∈X.

So μ (x * (x * y)) ≤ μ ((x * (x * y)) * z) ∨ μ (z), but X is a commutative BCI-algebra, then x * (x * y) = y * (y * (x * (x * y))). There for, μ (y * (y * x)) ≤ μ ((y * (y * (x * (x * y)))) * z) ∨ μ (z). This shows that μ is doubt fuzzy sub-commutative ideal of X. By applying proposition(3.8) and lemma(3.4), we have:

Theorem 3.9.

If X is a commutative BCI-algebra, then a fuzzy set μ of X is a doubt fuzzy ideal of X if and only if it is a doubt fuzzy sub-commutative ideal of X.

Theorem 3.10.

A fuzzy set μ of a BCI-algebra **X** is a fuzzy sub-commutative ideal of X if and only if its complement μ^c is a doubt fuzzy sub-commutative ideal of X.

Proof.

Let μ be a fuzzy sub-commutative ideal of a BCI-algebra X and let x, y, z ∈X. Then

μ^c (0) = 1 − μ (0) ≤ 1 − μ (x) = μ^c (x) and μ^c (x∗(x∗y)) = 1 − μ (x∗(x∗y)) ≤

1 − (μ ((y∗(y∗(x∗(x∗y))))∗z) ∧ μ (z)) =

1− [(1− μ^c ((y∗(y∗(x∗(x∗y))))∗z)) ∧(1− μ^c (z))]

= μ^c ((y * (y * (x * (x * y)))) * z) ∨ μ^c (z).

So, μ^c is a doubt fuzzy sub-commutative ideal of X.

Now let μ^c be a doubt fuzzy sub-commutative ideal of X, and let x, y, z ∈ X. Then

μ (0) = 1− μ^c (0) ≥ 1 − μ^c (x) = μ (x) , and

μ (x* (x * y)) = 1− μ^c (x * (x * y))

≥ 1 − [(1− μ ((y*(y*(x*(x*y))))*z)) ∨ (1− μ (z))]

= μ ((y∗(y∗(x∗(x∗y)))) ∗ z) ∧ μ (z).

Thus, μ is a fuzzy sub-commutative ideal of X.

Definition 3.11. ([5])

A fuzzy set μ in X is called doubt fuzzy p-ideal of X if it satisfies

(DF_1) μ (0) ≤ μ (x),

(DF_4) μ (x) ≤ μ ((x * z) * (y *z)) ∨ μ (y),

for all x, y, z ∈ X.

Remark(1)

Every doubt fuzzy p-ideal is doubt fuzzy ideal, but the converse does not hold.

Remark(2)

Take z = x and y = 0 in (DF_4), then every doubt fuzzy p-ideal in X satisfies the inequality

μ (x) ≤ μ (0 * (0 * x)) for all x ∈X.

Theorem 3.12.

Every doubt fuzzy p-ideal of X is doubt fuzzy sub-commutative ideal of X.

Proof.

Let μ be a doubt fuzzy p-ideal of X. Then μ is a doubt fuzzy ideal of X and

(0*(0*(x*(x *y))))*(y*(y*(x*(x*y))))

= (0*(y*(y*(x*(x*y)))))*(0*(x*(x*y))) [by(1)]

= ((0*y) *(0*(y*(x *(x *y))))))*(0*(x*(x*y))) [by(5)]

= ((0*y)*((0*y)*(0*(x*(x *y))))))* (0 * (x*(x * y)))

= ((0*y)*(0*(x*(x *y))))*((0*y)*(0*(x*(x *y)))) = 0. [by(1)]

From remark(2) we have,

μ (x *(x * y)) ≤ μ (0 * (0 * (x *(x * y)))), but 0*(0*(x∗(x*y))) ≤ y*(y*(x*(x*y))). Since every doubt fuzzy ideal is order preserving, then

μ (0*(0*(x*(x*y)))) ≤ μ (y*(y*(x*(x*y)))) , hence μ (x*(x*y)) ≤ μ (y*(y*(x*(x*y)))). From theorem 3.7, we get μ is doubt fuzzy sub-commutative ideal of X.

In the following example, we see that the converse of theorem 3.12 may not be tru

Example 3.13. Consider a BCI-algebra X = {0,a,1,2,3} with Cayley table

Define an anti fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.2$, $\mu(a) = 0.5$ and $\mu(1) = \mu(2) = \mu(3) = 0.7$. Then μ is a fuzzy ideal of X in which the inequality $\mu(x*(x*y)) \leq \mu(y*(y*(x*(x*y))))$ holds for all x, y $\in$ X. Using theorem 3.7, we see that μ is a doubt fuzzy sub-commutative ideal of X. μ is not doubt fuzzy p-ideal of X because $\mu(a) > \mu((a*1)*(0*1)) \vee \mu(0)$.

Theorem 3.14.
For any doubt fuzzy sub-commutative ideal μ of X, the set $X_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$ is sub-commutative ideal of X.

Proof. Clearly $0 \in X_\mu$. Let x, y, z $\in$ X be such that $(y*(y*(x*(x*y))))*z \in X_\mu$ and $z \in X_\mu$.
By (DF_3), we have
$\mu(x*(x*y)) \leq \mu((y*(y*(x*(x*y))))*z) \vee \mu(z) = \mu(0)$, which implies from ($DF_1$) that
$\mu(x*(x*y)) = \mu(0)$. Then $x*(x*y) \in X_\mu$ and X_μ is a sub-commutative ideal of X.

Applying Theorems 3.12 and 3.14, we have the following corollary.

Corollary 3.15.
If μ is a doubt fuzzy p-ideal of X, then the set $X_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$ is a sub- commutative ideal of X.

Theorem 3.16.
Let μ be a fuzzy set of BCI-algebra X. Then μ is a doubt fuzzy sub-commutative ideal of X if and only if for each $t \in [0,1]$, $t \geq \mu(0)$, the lower t-level cut μ^t is a sub-commutative ideal of X.

Proof.
Let μ be a doubt fuzzy sub-commutative ideal of X and let $t \in [0,1]$ with $\mu(0) \leq t$, by (DF_1), we have $\mu(0) \leq \mu(x)$ for all $x \in X$. But $\mu(x) \leq t$ for all $x \in \mu^t$ and so $0 \in \mu^t$. Let x , y, x $\in$ X be such that $(y*(y*(x*(x*y))))*z \in \mu^t$ and $z \in \mu^t$, then $\mu(z) \leq t$ and $\mu((y*(y*(x*(x*y))))*z) \leq t$. Since μ is doubt fuzzy sub-commutative ideal, it follow that

$\mu(x*(x*y))$

*	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

$\leq \mu((y*(y*(x*(x*y))))*z) \vee \mu(z)\} \leq t$, then $x*(x*y) \in \mu^t$. Therefore μ^t is sub-commutative ideal of X.

Conversely, let μ^t be a sub-commutative ideal of X. We only need to show that (DF_1), (DF_3) are true.
If (DF_1) is false, then there exist $x_0 \in X$ such that $\mu(0) > \mu(x_0)$. If we take $t_0 = \frac{1}{2}\{\mu(0) + \mu(x_0)\}$, then $\mu(0) > t_0$ and $0 \leqslant \mu(x_0) < t_0 \leqslant 1$, hence $x_0 \in \mu^{t_0}$ and $\mu^{t_0} \neq \phi$.But μ^{t_0} is sub-commutative ideal of X, then $0 \in \mu^{t_0}$ and so $\mu(0) \leq t_0$, contradiction, hence $\mu(0) \leq \mu(x)$ for all $x \in X$.

Now, assume (DF_3) is not true, then there exist $x_0, y_0, z_0 \in X$ such that
$\mu(x_0*(x_0*y_0)) > \mu((y_0*(y_0*(x_0*(x_0*y_0))))*z_0) \vee \mu(z_0)$. Putting
$s_0 = \frac{1}{2}\{\mu(x_0*(x_0*y_0)) + [\mu((y_0*(y_0*(x_0*(x_0*y_0))))*z_0) \vee \mu(z_0)]\}$, then $s_0 < \mu(x_0*(x_0*y_0))$, and
$0 \leq \mu((y_0*(y_0*(x_0*(x_0*y_0))))*z_0) \vee \mu(z_0) < s_0 \leq 1$.

Thus we have
$\mu((y_0*(y_0*(x_0*(x_0*y_0))))*z_0) < s_0$, $\mu(z_0) < s_0$.
Which imply that
$(y_0*(y_0*(x_0*(x_0*y_0))))*z_0 \in \mu^{s_0}$ and $z_0 \in \mu^{s_0}$, but μ^{s_0} is an sub-commutative ideal of X, thus $x_0*(x_0*y_0) \in \mu^{s_0}$ or $\mu(x_0*(x_0*y_0)) \leq s_0$.
This a contradiction, ending the proof.

Theorem 3.17.
If μ is a doubt fuzzy sub-commutative ideal of a BCI-algebra X. then μ_{μ^t} is also a doubt fuzzy sub-commutative ideal of X where $t \in [0,1]$, and $t \geq \mu(0)$.

Proof.
From theorem 3.16, it is sufficient to show that $(\mu_{\mu^t})^\delta$ is a sub-commutative ideal of X , where $\delta \in [0,1]$ and $\delta \geq \mu_{\mu^t}(0)$.Clearly , $0 \in (\mu_{\mu^t})^\delta$.
Let x, y, z $\in$ X be such that
$(y*(y*(x*(x*y))))*z \in (\mu_{\mu^t})^\delta$ and $z \in (\mu_{\mu^t})^\delta$. Thus
$\mu_{\mu^t}((y*(y*(x*(x*y))))*z) \leq \delta$ and $\mu_{\mu^t}(z) \leq \delta$.

We claim that
$x*(x*y) \in (\mu_{\mu^t})^{\delta}$ or $\mu_{\mu^t}(x*(x*y)) \le \delta$. If $(y*(y*(x*(x*y))))*z \in \mu^t$ and $z \in \mu^t$, then $x*(x*y) \in \mu^t$, because μ^t is a sub-commutative ideal of X. and hence
$\mu_{\mu^t}(x*(x*y)) = \mu(x*(x*y)) \le \mu((y*(y*(x*(x*y))))*z) \vee \mu(z) = \mu_{\mu^t}((y*(y*(x*(x*y))))*z) \vee \mu_{\mu^t}(z) \le \delta$
and so $x*(x*y) \in (\mu_{\mu^t})^{\delta}$.

If $(y*(y*(x*(x*y))))*z \notin \mu^t$ or $z \notin \mu^t$, then $\mu_{\mu^t}((y*(y*(x*(x*y))))*z) = 0$ or $\mu_{\mu^t}(z) = 0$, and so $\mu_{\mu^t}(x*(x*y)) \le \delta$ and so $x*(x*y) \in (\mu_{\mu^t})^{\delta}$. Therefor $(\mu_{\mu^t})^{\delta}$ is a sub-commutative ideal of X.

4. Homomorphism of doubt fuzzy sub-commutative ideal of BCI-algebra

Definition 4.1.
Let f be a mapping of BCI-algebra X into BCI-algebra Y and $A \subseteq X$, $B \subseteq Y$. The image of A in Y is $f(A) = \{f(a) \mid a \in A\}$ and the inverse image of B is $f^{-1}(B) = \{g \in X \mid f(g) \in B\}$.

Definition 4.2.
Let $(X, *, 0)$ and $(Y, *^{\backslash}, 0^{\backslash})$ be a BCI-algebras. A mapping $f: X \to Y$ is said to be a homomorphism if $f(x*y) = f(x) *^{\backslash} f(y)$ for all $x, y \in X$.

Theorem 4.3.
Let f be a homomorphism of BCI-algebra X into a BCI-algebra Y, then:
(i) If 0 is the identity in X, then $f(0)$ is the identity in Y.
(ii) If A is sub-commutative ideal of X, then $f(A)$ is sub-commutative ideal of Y.
(iii) If B is sub-commutative ideal of Y, then $f^{-1}(B)$ is sub-commutative ideal of Y.
(vi) If X is commutative BCI-algebra, then $\ker f$ is sub-commutative ideal of X.

Proof.
(i) By using Definition 2.1 and Definition 4.2, we have $f(0) = f(0*0) = f(0) *^{\backslash} f(0) = 0^{\backslash}$.
(ii) Let A be an sub-commutative ideal of X. Clearly $0^{\backslash} \in f(A)$. If $(f(y) *^{\backslash}(f(y) *^{\backslash}(f(x) *^{\backslash}(f(x) *^{\backslash} f(y))))) *^{\backslash} f(z) \in f(A)$
and $f(z) \in f(A)$, then
$f((y*(y*(x*(x*y))))*z) \in f(A)$, since f is a homomorphism, we have $(y*(y*(x*(x*y))))*z \in A$ and $z \in A$. Since A is sub-commutative ideal, then $x*(x*y) \in A$, and hence
$f(x*(x*y)) = f(x) *^{\backslash}(f(x) *^{\backslash} f(y)) \in f(A)$.
Then $f(A)$ is sub-commutative ideal of Y.
(iii) Let B be an sub-commutative ideal of $f(X)$, since $f(0) = 0^{\backslash}$, $0 \in f^{-1}(B)$.
Let $(y*(y*(x*(x*y))))*z \in f^{-1}(B)$, $z \in f^{-1}(B)$ for all $x, y, z \in X$, then
$f((y*(y*(x*(x*y))))*z) \in B$, $f(z) \in B$, but f is homomorphism, then
$(f(y) *^{\backslash}(f(y) *^{\backslash}(f(x) *^{\backslash}(f(x) *^{\backslash} f(y))))) *^{\backslash} f(z) \in B$
and $f(z) \in B$, since B is sub-commutative ideal, we have
$f(x) *^{\backslash}(f(x) *^{\backslash} f(y)) = f(x*(x*y)) \in B$. Hence $x*(x*y) \in f^{-1}(B)$, then $f^{-1}(B)$ is sub-commutative ideal.
(iv) Let $x, y, z \in X$ such that
$(y*(y*(x*(x*y))))*z \in \ker f$, $z \in \ker f$, then
$f((y*(y*(x*(x*y))))*z) = 0^{\backslash}$, $f(z) = 0^{\backslash}$, since f is homomorphism we have
$(f(y) *^{\backslash}(f(y) *^{\backslash}(f(x) *^{\backslash}(f(x) *^{\backslash} f(y))))) *^{\backslash} f(z) = 0^{\backslash}$.
Then,
$(f(y) *^{\backslash}(f(y) *^{\backslash}(f(x) *^{\backslash}(f(x) *^{\backslash} f(y))))) *^{\backslash} 0^{\backslash}$
$= (f(y) *^{\backslash}(f(y) *^{\backslash}(f(x) *^{\backslash}(f(x) *^{\backslash} f(y)))))$
$= f((y*(y*(x*(x*y))))) = f(x*(x*y)) = 0^{\backslash}$, by using theorem 2.5, then we have $x*(x*y) \in \ker f$. So $\ker f$ is sub-commutative ideal of X.

Definition 4.4.
Let $f: X \to Y$ be a homomorphism of BCI-algebras and β be a fuzzy set of Y, then β^f is called the pre-image of β under f and its denoted by
$\beta^f(x) = \beta(f(x))$, for all $x \in X$.

Theorem 4.5.
Let $f: X \to Y$ be a homomorphism of BCI-algebras. If β is a doubt fuzzy sub-commutative ideal of Y, then β^f is a doubt fuzzy sub-commutative ideal of X.

Proof.
Since β is a doubt fuzzy sub-commutative ideal of

Y, then $\beta(0') \leq \beta(f(x))$ for every x $\in$ X. So $\beta^f(0) = \beta(f(0)) = \beta(0') \leq \beta(f(x)) = \beta^f(x)$.
For any x , y , z $\in$ X, we have
$\beta^f(x*(x*y)) = \beta(f(x*(x*y)) =$
$\beta(f(x) *' (f(x) *' f(y))) \leq$
$\beta((f(y) *' (f(y) *' (f(x) *' (f(x) *' f(y))))) *' f(z)) \vee \beta(f(z))$
$= \beta(f((y*(y*(x*(x*y))))*z)) \vee \beta(f(z))$
$= \beta^f((y*(y*(x*(x*y))))*z) \vee \beta^f(z)$.
Then β^f is a doubt fuzzy sub-commutative ideal of X.

Theorem 4.6.
Let f : X $\rightarrow$ Y be an epimorphism of BCI-algebras.If β^f is a doubt fuzzy sub-commutative ideal of X, then β is a doubt fuzzy sub-commutative ideal of Y.

Proof.
Let β^f be a doubt fuzzy sub-commutative ideal of X and y $\in$ Y, there exist x $\in$ X such that $f(x) = y$, then $\beta(y) = \beta(f(x)) = \beta^f(x) \leq \beta^f(0) = \beta(f(0)) = \beta(0')$. Let $x', y', z' \in Y$, then there exist x , y , z $\in$ X such that $f(x) = x'$, $f(y) = y'$ and $f(z) = z'$. It follows that
$\beta(x' *' (x' *' y')) = \beta(f(x) *' (f(x) *' f(y)))$
$= \beta(f(x*(x*y)) = \beta^f(x*(x*y))$
$\leq \beta^f((y*(y*(x*(x*y))))*z) \vee \beta^f(z)$
$= \beta(f((y*(y*(x*(x*y))))*z)) \vee \beta(f(z))$
$= \beta((f(y) *' (f(y) *' (f(x) *' (f(x) *' f(y))))) *' f(z)) \vee \beta(f(z))$
$= \beta((y' *' (y' *' (x' *' (x' *' y')))) *' z') \vee \beta(z')$
and hence β is a doubt fuzzy sub-commutative ideal of Y.

5. Cartesian product of DFSC-ideals

Definition 5.1. ([1]) .
A fuzzy relation on any set X is a fuzzy subset $\mu : X \times X \rightarrow [0,1]$.

Definition 5.2.
If μ is a fuzzy relation on a set X and β is a fuzzy subset of X, then μ is a doubt fuzzy relation on β if $\mu(x , y) \geq \beta(x) \vee \beta(y)$ for all x , y $\in$ X.

Definition 5.3.
Let μ and λ be doubt fuzzy subsets of a set X. The Cartesian product $\mu \times \lambda : X \times X \rightarrow [0,1]$ is defined by $(\mu \times \lambda)(x , y) = \mu(x) \vee \lambda(y)$
for all x , y $\in$ X.

Iemma 5.4.([1])
Let μ and λ be fuzzy subsets of a set X.
Then ,
(i) $\mu \times \lambda$ is a fuzzy relation on X,
(ii) $(\mu \times \lambda)_t = \mu_t \times \lambda_t$ for all t$\in$[0,1].

Definition 5.5.
If β is a fuzzy subset of a set X, the strongest doubt fuzzy relation on X (that is a doubt fuzzy relation on β) is μ_β, given by $\mu_\beta(x , y) = \beta(x) \vee \beta(y)$
for all x , y $\in$ X.

Proposition 5.6. For a given fuzzy subset β of a BCI-algebra X, let μ_β be the strongest doubt fuzzy relation on X. If μ_β is a doubt fuzzy sub-commutative ideal of X × X, then $\beta(x) \geq \beta(0)$ for all x $\in$ X.

Proof. Since μ_β is doubt fuzzy sub-commutative ideal of X × X, then from (DF_1) that
$\mu_\beta(x,x) = \beta(x) \vee \beta(x) \geq \mu_\beta(0,0) = \beta(0) \vee \beta(0)$
where (0,0) $\in$ X ×X, then $\beta(x) \geq \beta(0)$.

<u>**Remark 5.7.**</u>
Let X and Y be BCI-algebras, we define * on X × Y by,for every (x , y) , (u , v) $\in$ X × Y,
(x ,y)*(u ,v) = (x*u , y*v).
Then clearly(X × Y ; * , (0 , 0)) is a BCI-algebra.

Theorem 5.8.
Let μ and β be doubt fuzzy sub-commutative ideals of BCI-algebra X. Then $\mu \times \beta$ is a doubt fuzzy sub-commutative ideal of X × X.

Proof.
Let μ and β be doubt fuzzy sub-commutative ideals of BCI-algebra X, for every (x , y) $\in$ X × X, we have $(\mu \times \beta)(0, 0) = \mu(0) \vee \beta(0) \leq \mu(x) \vee \beta(y)$
$= (\mu \times \beta)(x , y)$.
Now we let (x_1,x_2) , (y_1,y_2) , (z_1,z_2) $\in$ X × X, then we have $(\mu \times \beta)((x_1 , x_2) *((x_1 , x_2) *(y_1 , y_2))) =$
$(\mu \times \beta)(x_1 *(x_1 * y_1) , x_2 *(x_2 * y_2)) =$
$\mu(x_1 * (x_1 * y_1)) \vee \beta(x_2*(x_2*y_2)) \leq$
$\{\mu((y_1*(y_1*(x_1*(x_1*y_1))))*z_1) \vee \mu(z_1)\} \vee$
$\{\beta((y_2 * (y_2 * (x_2 * (x_2 * y_2)))) * z_2) \vee \beta(z_2)\} =$
$\{\mu((y_1*(y_1*(x_1*(x_1*y_1))))*z_1) \vee \beta((y_2*(y_2*(x_2*(x_2*$

$y_2))))^*z_2) \vee \{\mu(z_1) \vee \beta(z_2)\} =$
$(\mu\times\beta)((y_1*(y_1*(x_1*(x_1*y_1))))*z_1,(y_2*(y_2*(x_2*(x_2*y_2))))*z_2) \vee (\mu\times\beta)(z_1, z_2)$
$= (\mu\times\beta)(((y_1,y_2)*((y_1,y_2)*((x_1,x_2)*((x_1,x_2)*(y_1,y_2)))))*(z_1,z_2)) \vee (\mu\times\beta)(z_1, z_2).$

Hence $\mu\times\beta$ is a doubt fuzzy sub-commutative ideal of X × X.

Analogous to theorem 3.2[10], we have a similar result for doubt fuzzy sub-commutative ideals, which can be proved in a similar manner, we state the result without proof.

Theorem 5.9.
Let μ and β be a fuzzy subsets of a BCI-algebra X such that $\mu\times\beta$ is a doubt fuzzy sub-commutative ideal of X × X. Then,

(i) Either $\mu(x) \geq \mu(0)$ or $\beta(x) \geq \beta(0)$ for all $x \in X$,

(ii) If $\mu(x) \geq \mu(0)$ for all $x \in X$, then either $\mu(x) \geq \beta(0)$ or $\beta(x) \geq \mu(0)$,

(iii) If $\beta(x) \geq \beta(0)$ for all $x \in X$, then either $\mu(x) \geq \mu(0)$ or $\beta(x) \geq \mu(0)$,

(iv) Either μ or β is a doubt fuzzy sub-commutative ideal of X.

Theorem 5.10.
Let β be a fuzzy subset of a BCI-algebra X and let μ_β be the strongest doubt fuzzy relation on X. Then β is a doubt fuzzy sub-commutative ideal of X if and only if μ_β is doubt fuzzy sub-commutative ideal of X × X

Proof.
Assume that β is a doubt fuzzy sub-commutative ideal of X. We note from (DF_1) ,that $\mu_\beta(0,0) = \beta(0) \vee \beta(0) \leq \beta(x) \vee \beta(y) = \mu_\beta(x,y)$ for all $(x, y) \in X\times X$.

For all $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X\times X$, we get

$\mu_\beta((x_1, x_2)*((x_1,x_2)*(y_1, y_2)))$
$= \mu_\beta(x_1*(x_1*y_1), x_2*(x_2*y_2))$
$= \beta(x_1*(x_1*y_1)) \vee \beta(x_2*(x_2*y_2))\}$
$\leq \{\beta((y_1*(y_1*(x_1*(x_1*y_1))))*z_1) \vee \beta(z_1)\} \vee \{\beta((y_2*(y_2*(x_2*(x_2*y_2))))*z_2) \vee \beta(z_2)\}$
$= \{\beta((y_1*(y_1*(x_1*(x_1*y_1))))*z_1) \vee \beta((y_2*(y_2*(x_2*(x_2*y_2))))*z_2) \vee \{\beta(z_1) \vee \beta(z_2)\}$
$= \mu_\beta((y_1*(y_1*(x_1*(x_1*y_1))))*z_1,(y_2*(y_2*(x_2*(x_2*y_2))))*z_2) \vee \mu_\beta(z_1, z_2)\}$
$= \mu_\beta(((y_1,y_2)*((y_1,y_2)*((x_1,x_2)*((x_1,x_2)*(y_1,y_2)))))*(z_1,z_2)) \vee \mu_\beta(z_1, z_2)\}$.Hence, μ_β is a doubt fuzzy sub-commutative ideal of X × X. Conversely, suppose that μ_β is a doubt fuzzy sub-commutative ideal of X × X. Then for all $(x, y) \in X\times X$, $\beta(0) \vee \beta(0) = \mu_\beta(0, 0) \leq \mu_\beta(x, y) = \beta(x) \vee \beta(y)$, it follows that $\beta(0) \leq \beta(x)$ for all $x \in X$, which proves (DF_1).
Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X\times X$. Then,

$\beta(x_1*(x_1*y_1)) \vee \beta(x_2*(x_2*y_2))$
$= \mu_\beta(x_1*(x_1*y_1), x_2*(x_2*y_2))$
$= \mu_\beta((x_1, x_2)*((x_1, x_2)*(y_1, y_2)))$
$\leq \mu_\beta(((y_1,y_2)*((y_1,y_2)*((x_1,x_2)*((x_1,x_2)*(y_1,y_2)))))*(z_1,z_2)) \vee \mu_\beta(z_1, z_2)$
$= \mu_\beta((y_1*(y_1*(x_1*(x_1*y_1))))*z_1, (y_2*(y_2*(x_2*(x_2*y_2))))*z_2) \vee \mu_\beta(z_1, z_2)$
$= \{\beta((y_1*(y_1*(x_1*(x_1*y_1))))*z_1) \vee \beta((y_2*(y_2*(x_2*(x_2*y_2))))*z_2)\} \vee \{\beta(z_1) \vee \beta(z_2)\}$
$= \{\beta((y_1*(y_1*(x_1*(x_1*y_1))))*z_1) \vee \beta(z_1)\} \vee \{\beta((y_2*(y_2*(x_2*(x_2*y_2))))*z_2)\} \vee \beta(z_2)\}$.

Take $x_2 = y_2 = z_2 = 0$, then

$\beta(x_1*(x_1*y_1)) \leq \beta((y_1*(y_1*(x_1*(x_1*y_1))))*z_1) \vee \beta(z_1).$

Then β is a doubt fuzzy sub-commutative ideal of X.

Corresponding author
Ahmed I. Marie
Department of Mathematic, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt
samymostafa@yahoo.com
ahmedibrahim500@yahoo.com

References

[1] p. Bhattacharya and N.P. Mukherjee, Fuzzy relations and fuzzy group, Inform. Sci. 36(1985) 267 – 282.

[2] S. M. Hong and Y. B. Jun, Anti fuzzy ideals in BCK-algebras, Kyung. Pook. Math. J. 38(1998), 145 – 150.

[3] Sabahattim Ilbira, Alev Firat and Y. B. Jun, On symmetric Bi-derivations of BCI- algebras, Applied Mathematical Science. 5(2011) no. 2957 – 2966.

[4] K. Iseki and S. Tanaka, An introduction to theory of BCK-algebras, Math. Japon, 23(1978),1 – 26.

[5] Zhan Jianming and Tan Zhisong, Doubt FuzzyBCI-algebras,IJMMS30:1(2002) 49-56.

[6]Y. B. Jun and E. H. Roh, Fuzzy

commutative ideals of BCK-algebra, Fuzzy Sets and Systems 64(1994) 401 – 405.

[7] Y. B. Jun, Doubt fuzzy BCK/BCI-algebras, Soochow J. Math. 20(1994),no.3,351–358.

[8] Y. L. Liu and J. Meng, Sub-implicative ideals and sub-commutative ideals of BCI-algebras, Soochow J. Math, 26 (2000), 441- 453.

[9] Y. L. Liu, San Yang Liu and J. I. Meng, FSI-ideals and FSC-ideals of BCI-algebras, Bull Korean Math. 41(2004), No.1,pp. 167 – 179.

11/11/2011

[10] J. Meng and X. L. Xin, commutative BCI-algebras, Math.Japon,37(1992), 569 – 572.

[11] J. Meng and Xiu-e Guo, On fuzzy ideals in BCK/BCI-algebras, Fuzzy Sets and Systems,149(2005), 509 – 525.

[12] Samy M. Mostafa, Fuzzy implicative ideals in BCK-algebras, Fuzzy Sets and Systems, 87(1997) 361 – 368.

[13] O. G. Xi, Fuzzy BCK-algebras, Math. Japon, 36(1991), 935 – 942.

[14] L. A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965), 338–353.