Directional Extension of the Domain of Attraction to Increase Critical clearing time of nonlinear systems

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Abstract: In this paper a new approach for directional enlargement of domain of attraction based on design of the operating equilibrium of nonlinear dynamic systems and determining optimal values of controlling parameters is proposed. The method estimates domain of attraction with parameter dependant ellipses using quadratic Lyapunov functions, and finds these parameters such that they enlarge the estimated attraction region along the direction of interest. The problem of attraction region enlargement is defined in the form of a novel bi level optimization problem that focuses on extending elliptic area along fault running vector. In addition we show that the proposed method can effectively be applied for increasing critical clearing time of nonlinear systems. This application leads to the increase the maximum allowable time for removing the fault of nonlinear systems and the reduction the control cost. The efficiency of the proposed method is shown in the simulation part by some examples.
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1. Introduction

In this paper, a new method for directional enlargement (*DE*) of domain of attraction is proposed. In systems with large domain of attraction, disturbance appearance is not a serious problem and its affect is usually removed with systematic controllers. On the other hand, in systems with bounded domain of attraction increasing critical fault clearing time, the maximum allowable time to clear the fault and keep the system stable are very important.

Determining the Domain of Attraction (*DA*) of a stable equilibrium point, which is the set of all initial conditions from which the states converge to that point, is an important problem in nonlinear systems theory. In general *DA* cannot be exactly calculated. Different methods have been proposed to estimate the *DA*. These methods can be classified in two general groups, Lyapunov based and non-Lyapunov based. The first group contains two main steps [1-5].

- A suitable Lyapunov function (*LF*) is suggested based on the structure of the system.
- *DA* is estimated based on the suggested *LF*.

In [6] the Linear Matrix Inequalities (*LMI*) optimization is used to estimate *DA*. Some papers propose the idea of estimating *DA* with the union of the level sets of some Lyapunov functions instead of using just one of them [7].

Lyapunov based methods usually confront with some limitations such as depending the size of estimated *DA* on the chosen *LF* and usage of usual optimization tools such as *LMI* or moment matrices which are usually applicable just for polynomial *LF* s[8]. In spite of these constraints, the simplicity of these methods lead to the wide applications of them in estimating the *DA*. *DA* enlargement is a problem which is closely related to *DA* estimation.

In most of the papers, *DA* enlarged in all directions (global enlargement) and the method for changing the form of *DA* to a designed one has not presented yet [9-13]. The purpose of this paper is the design of nonlinear systems with specific focus on the directional extension of the *DA*. We also show that this method can effectively used for optimal increasing critical fault clearing time.

Lower control cost, simpler and faster determination of control parameters values and application for increasing clearing time are advantages of this method in compare with global enlarging method.

This work contains four sections. In Section 2 introduces basic definitions and theorems used along the paper. Section 3 presents the proposed approach and the main problem. In Section 4 the developed methodology is applied on some illustrative examples. Section 5 concludes the paper.

2. Background definitions and theorems

Consider the following nonlinear system. Let x_0 be an (isolated) asymptotic equilibrium point of this system:

$$\dot{x} = f(x)\ ,\ x \in R^n\ ,\quad x(t_0) = x_0$$
(1)

Where $f(x)$ is a nonlinear function which satisfies the lipshitz conditions. For such system the following definitions and theorems are proposed :

2.1. Definitions

Definition 1 (Equilibrium Point [14]). A point $x_e \in R^n$ is called an equilibrium point of system (1) if $f(x_e) = 0$. The equilibrium points of system (1) correspond to the intersection of the nullclines of the system, meaning the curves given by $f(x) = 0$.

In the sequel, without loss of generality, we assume that the equilibrium point under study coincides with the origin of the state space of R^n ,$(x_e = 0)$.

Definition 2 (Asymptotic stability [15]). The origin is called asymptotically stable if,

- It is stable.
- There exists a $\mu > 0$ having the property $\lim_{t\to\infty} x(t, x_0) = 0$ whenever $\|x_0\| < \mu$. (2)

Definition 3 (Positive and negative (semi) definite functions [15,16]). Let $D \subseteq R^n$, $0 \in D$. A function $V(x) : D \to R$ is called positive definite (positive semidefinite) on D if

$V(0) = 0$ and $V(x) > 0$ $(V(x) \geq 0)$ $\forall x \in D \setminus \{0\}$.
(3)

V(x) is called negative definite (negative semidefinite) if -V(x) is positive definite (positive semidefinite).

Definition 4 (Domain of attraction [14]). The domain of attraction of the origin is given by

$$DA = \{x_0 \in R^n | \lim_{t\to\infty} x(t, x_0) = 0\}. \tag{4}$$

Definition 5 (Lyapunov function [15,16]). Let V(x) be a continuously differentiable real-valued function defined on a domain $D \subseteq R^n$ containing the origin. The function $V(x)$ is called a Lyapunov function for the dynamical system (1) if the following conditions are fulfilled:

• $V(x)$ is positive definite on D.
• The time derivative of $V(x)$ along the trajectories of (1)

$$\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right)^T f(x) \tag{5}$$

is negative definite on D.

Definition 6 (Fault running vector). Fault running vector is a normal vector along the line which connects equilibrium point to the contact point of faulted system trajectory and the DA closure of pre-fault system. Faulted system is a system with some fault occurrences such as short circuit which happens in a power system and pre-fault system is the system before fault happens.

Definition 7 (Critical clearing time). Critical time is the maximum allowable time to clear the fault and keep the system stable.

Definition 8 (Directional enlargement of *DA*). Considering the following system with equilibrium state $x_e = 0$. Directional enlargement of *DA* is determines an optimal value for controlling parameters $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_h]^T$ to extend the domain of attraction of the system along a desired direction like $e = [e_1, e_2, \dots, e_n]^T$.

$$\dot{x} = f(x, \alpha) \qquad f : R^n \times R^{+h} \to R^m \tag{6}$$

In this paper fault running vector is considered as the desired direction.

2.2. Theorems

Proposition 1 (Directional enlargement coefficient). Directional enlargement coefficient can be presented in the form of an optimization problem as follows:

$$\begin{aligned} &\underset{\gamma \in R^+}{maximize} \quad |\gamma| \\ &\text{S.t.} \quad : \dot{x} = f(x, t) \\ &\qquad\quad x(0) = \gamma\, e \\ &\qquad\quad \lim_{t\to\infty} x(t) = x_0 \end{aligned} \tag{7}$$

Proof : Let γ^* be the optimal value of γ, then $\gamma^* e$ will be the farthest stable initial state from x_e along fault running vector e .
□

In this paper we define γ by *DE* coefficient. *DE* coefficient can be a proper criterion to compare the efficiency of *DE* methods on enlarging the *DA* along e.

Theorem 1 (Local asymptotic stability (Lyapunov) [16]). If there exists in some neighborhood δ of the origin a Lyapunov function $V(x)$ such that $\dot{V}(x)$ is negative definite on δ , then the origin is asymptotically stable.

Theorem 2 (Jacobian's eigenvalues and local asymptotic stability [15]). Let $A = \partial f / \partial x(x)\|_{x=0}$ be the Jacobian of system (1) at the origin then

- The origin is asymptotically stable if all eigenvalues of A have negative real parts.
- The origin is unstable if one or more eigenvalues of A have positive real parts.

Theorem 3 (Guaranteed estimation of the domain of attraction [16]**).** Let $V(x)$ be a Lyapunov function for system (1) in the domain

$$\Omega_c = \{x \| V(x) \le c\}, \qquad c > 0. \tag{8}$$

Assume, moreover, that Ω_c is bounded and contains the origin. If $\dot{V}(x)$ is negative definite in Ω_c, then the origin is asymptotically stable and every solution in Ω_c tends to the origin as t converges to infinity.

3. Proposed approach
3.1. Preliminaries

Lemma 1 (Finding *DE* coefficient from elliptic estimated *DAs*) : Consider $V = x^T P x$, $P = P^T \epsilon R^{n \times n}$, $P > 0$ as a quadratic Lyapaunov function. Let $e = [\cos\varphi_e, \sin\varphi_e]^T$ and x can be defined as $x = re = [r\cos\varphi_e, r\sin\varphi_e]$. It is clear that $V(x) = 1$ results in $r^2 = \frac{1}{e^T P e}$.

If $r^* = \max r$, the answer of following optimization problem, is an *DE* coefficient along e vector.

$$\max_{P, r} \; r$$

$$s.t. \quad r - 1/\sqrt{e^T P e} = 0 \tag{9a}$$

$$\dot{V} = x^T P \dot{x} + \dot{x}^T P x < 0 \tag{9b}$$

$$\dot{x} - f(x, \alpha) = 0 \tag{9c}$$

$$e = [\cos\varphi_e, \sin\varphi_e]^T \tag{9d}$$

Where $e = [\cos\varphi_e \; \sin\varphi_e]^T$, demonstrates the fault running vector.

Proof : Let $x = [r\cos\varphi_e \; r\sin\varphi_e]^T$, according to theorem 2 , (9a) and (9d) lead to $r^2 e^T P e = 1$ as an elliptic estimate of *DA*. The above optimization problem finds optimal controlling parameters to extend the estimated *DA* along fault running vector (not along all directions) so r^* which is the distance of the contact point of fault running vector with closure of the best estimate of *DA* along this vector , can be represented as an *ED* coefficient. □

In order to clarify Lemma1, consider Figure 1, the interior sets of $V_1 = 1$ and $V_2 = 1$ are two elliptic estimates of *DA* obtained from two different quadratic Lyapunov functions V_1 and V_2 , $S = \{ r_k(\varphi_e) | V_k = r_k{}^2 e^T P_k e = 1, \; P_k = P_k{}^T \epsilon R^{n \times n}, \; P_k > 0 \;, \; \dot{V}_k < 0, \; k = 0,1, \dots \}$ is set of candidates which r^* which can be chosen from them.

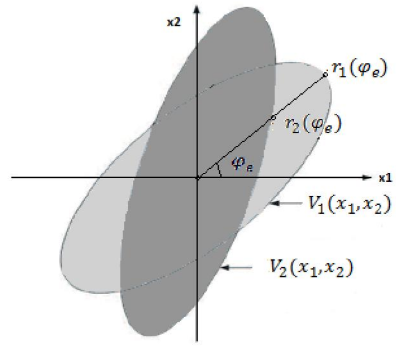


Figure 1. *DE* coefficient candidates

3.2. Main method

Although directional extension of *DA* is an interesting object in many applications such as power systems, finding a proper controlling vector to achieve this aim is not easy.

In the sequel, the domain of attraction of system (6) is estimated with a quadratic Lyapunov function which depends on controlling parameters vector. To enlarge the attraction domain along direction of interest, we consider the following optimal control problem. The purpose of this optimal control problem is finding the symmetric positive definite P matrix which leads to the elliptic *DA* around an stable equilibrium point x_e. This elliptic *DA* has the largest distance along the desired direction to stable.

Consider nonlinear system (1) with $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]^T$ as the cotrolling parameters vector. Let $[e_1, e_2, \dots, e_n]^T$ be the desired direction for *DA* enlargement and $x_e(\alpha)$ be an isolated equilibrium point of nonlinear system.

$$\underset{r, x_e, \alpha}{MAX} \; r$$

$$S.t. \quad f(x_e, \alpha) = 0 \tag{10a}$$

$$Re\{\lambda_i[A(x_e, \alpha)]\} < 0 \quad i = 1, \dots, n \tag{10b}$$

$$\underset{r, P}{Max} \; r \tag{10c}$$

$$S.t. \quad r - 1/\sqrt{e^T P e} = 0 \tag{10d}$$

$$\dot{x}^T P x + x^T P \dot{x} < 0 \tag{10e}$$

$$\dot{x} - f(x, \alpha) = 0 \tag{10f}$$

$$e = [\cos\varphi_e , \sin\varphi_e]^T \qquad (10g)$$

For the asymptotic stability constrains of nonlinear systems with equilibrium point x_e and α as the vector of controlling parameters, conditions (10a) and (10b) are considered. (10c) to (10f) indicate an optimization problem for to determine the *DE* coefficient (9) as described in Lemma1.

It should be noted that constraints (10c) to (10g) may have many local solutions. In order to avoid dummy solutions, problem (10c) to (10g) has to be solved to global optimality therefore in this contribution a standard implementation of a genetic algorithm adopted [17].

In order to solve optimization problem (10b) to (10g) a two level solution strategy is proposed. For each design variable is an equilibrium point x_e is calculated from (10a) and its feasibility with respect to constraint (10b) verified. If the equilibrium is asymptotically stable, problem (10c) to (10g) are solved at the ''inner level'' to find the ellipse which has the largest distance along the desired direction from stable equilibrium and its distance is returned as the objective function value.

4. Examples

4.1. Example 1

Consider the following nonlinear system:

$$\frac{dx_1}{dt} = -x_2$$
$$\frac{dx_2}{dt} = x_1 - \alpha(1 - x_1{}^2)x_2$$

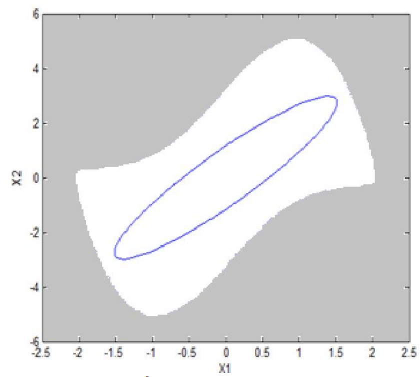
Where α is the controlling parameter and $\alpha \in h = [1 , 3]$. The analyzed equilibrium is the (0, 0) in h set.

At first $\varphi_e = 60^\circ$ is considered. For $\varphi_e = 60^\circ$ the best elliptic estimate of *DA* and the optimal value of controlling parameter $(\alpha = 2.9)$ is obtained from lemma2. In Figure 2(a) the elliptic estimate of *DA* and the actual *DA* for $\alpha = 2.9$ are illustrated.

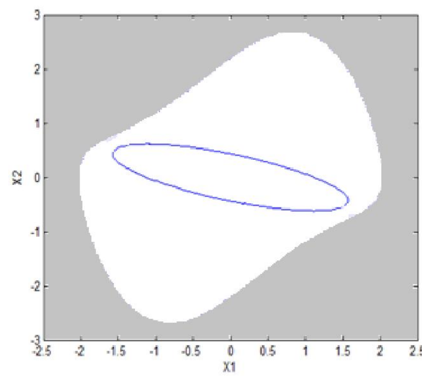
Figure 2(b) contains the result for $\varphi_e = 160^\circ$ and Figure 2(c) illustrates the result for $\varphi_e = 115^\circ$. Figure 3 indicates the efficiency of proposed method in increasing critical clearing time. It is considered that a fault happening results in the fault system trajectory is plotted in the form of a dashed curve.

According to definition 6 for such fault system, the fault running vector is $e = [\cos 95 \ \ \sin 95]$.

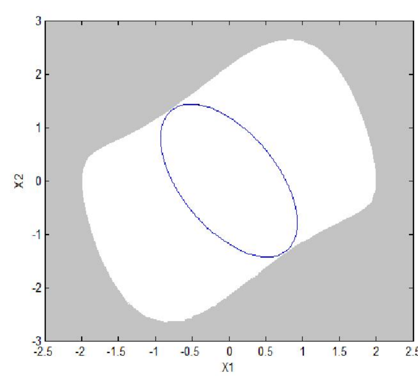
Table 1 shows *DE* coefficient for $\varphi_e = 60^\circ$, $\varphi_e = 160^\circ$ and $\varphi_e = 115^\circ$. Table 2 shows the numerical values of critical clearing time for $\alpha = 3$ and $\alpha = 1$.



(a) $\varphi_e = 60^\circ$, $\alpha = 2.9$



(b) $\varphi_e = 160^\circ$, $\alpha = 1.1$



(c) $\varphi_e = 115^\circ$, $\alpha = 1$

Figure 2. Results for Example1. elliptic estimate of *DA* (solid line), actual *DA* for optimal value of controlling parameter(white area)

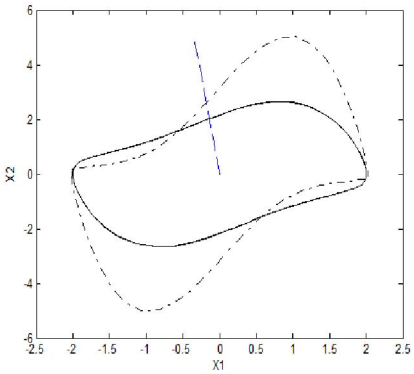


Figure 3. Results for Example 1. Increasing critical clearing time using Directional enlargement of *DA* for $\alpha = 1$ and $\alpha = 3$, $\varphi_\varepsilon = 95^\circ$,
(.-.-) trajectory of system for nominal $\alpha = 1$, (__) trajectory of system for optimal $\alpha = 3$, (--) trajectory fault system after fault.

Table 1. *DE* coefficients for Example 1.

Desired direction	α	*DE* coefficient
60 °	2.9	3.0190
160 °	1.1	1.5953
115°	1	1.5506

Table 2. numerical values of critical clearing time for Example 1.

case	α	Fault running vector	Critical clearing time
Optimal	3	$[\cos 95\,,\ \sin 95]$	0.958 s
Nominal	1	$[\cos 95\,,\ \sin 95]$	0.940 s

4.2. Example 2

Consider the following nonlinear system:

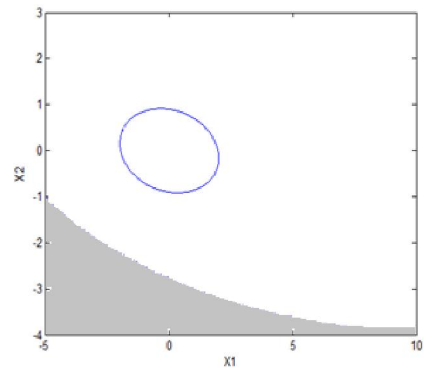
$$\frac{dx_1}{dt} = -0.84x_1 - 1.44x_2 - \alpha x_1 x_2$$
$$\frac{dx_2}{dt} = 0.54x_1 + 0.34x_2 + \alpha x_1 x_2$$

Where α is the controlling parameter and $\alpha \, \varepsilon \, h = [0.1\,, 0.9]$. The analyzed equilibrium is the (0, 0) in h set.

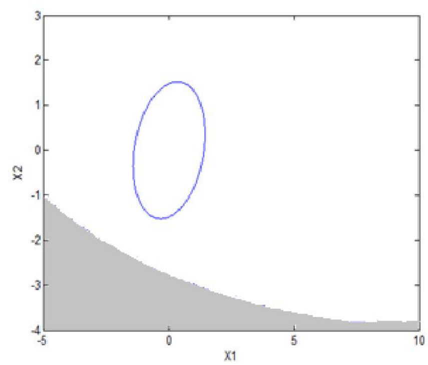
The best elliptic estimate of *DA* is obtained from Lemma2. The optimal estimate of DA and the actual *DA* for $\varphi_\varepsilon = 20^\circ$, $\varphi_\varepsilon = 50^\circ$ and $\varphi_\varepsilon = 120^\circ$ are plotted in Figures 4(a)-4(c) respectively.

According to Fig. 4, for this example, the optimal values of controlling parameters for enlarging *DA* along different directions are same ($\alpha = 0.1$). In the other words, the actual *DA* has uniform growth along different directions.

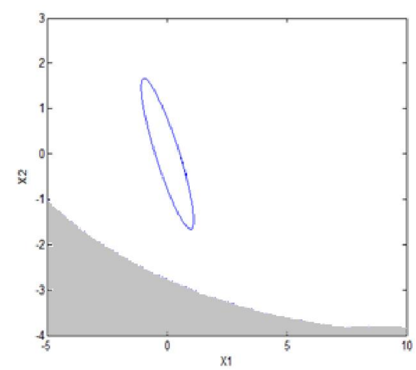
Table 2 contains *DE* coefficient for $\varphi_\varepsilon = 50^\circ$, $\varphi_\varepsilon = 120^\circ$ and $\varphi_\varepsilon = 20^\circ$ respectively.



(a)$\varphi_\varepsilon = 20^\circ$, $\alpha = 0.1$



(b) $\alpha = 50^\circ$, $\alpha = 0.1$



(c)$\varphi_\varepsilon = 120^\circ$, $\alpha = 0.1$

Figure. 4. elliptic estimate of *DA* (solid line), actual *DA* for optimal value of controlling parameter(white area)

Table3. *DE* coefficients for Example 2

Desired direction	α	*DE* coefficient
50 °	0.1	1.6487
120 °	0.1	1.9182
20 °	0.1	1.5189

5. Conclusions

In this paper a design approach to ensure asymptotic stability and enlarge *DA* of the resulting equilibrium point along desired direction is proposed.

The approach consists of a bi level optimization problem to maximize distance *DA* along desired direction and find optimal controller values. The first, we check for asymptotic stability of each potential equilibrium point. Then, distance *DA* along desired direction is calculated for each surviving candidate by solving an optimization sub-problem and its value is returned as the merit function for the outer problem.

It is illustrated that the proposed method can be effectively applied for increasing critical clearing time of nonlinear systems. This increase causes lower control cost and faster determination of controlling parameters.

Applying this method to increase critical clearing time of power systems is a part of our future work.

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