A nonlinear inverse problem with unknown radiation term

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Abstract: In this study we consider an inverse problem of linear heat equation with nonlinear boundary condition. We identify the temperature and the unknown radiation term from an overspecified condition on the boundary. At the beginning of the study, Taylor's series expansion is employed to linearize nonlinear term and then finite-difference method is used to discretize the problem domain. The least-squares method is adopted to modify the solution. To regularize the resultant ill-conditioned linear system of equations, we apply the Tikhonov regularization method to obtain the stable numerical approximation to the solution. Results show that an excellent estimation can be obtained within a couple of minutes CPU time at pentium IV-2.4 GHz PC.

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1. Introduction

The problem of determining unknown parameters in parabolic differential equations has bean treated by many authors [1-10]. Usually this problems involve the determination of a single unknown parameter from overspecified boundary data. In some applications, however, it is desirable to be able to determine more than one parameter from the given boundary data [2,6]. It is well known that the radiative heat is a function of temperature. In certain radiative heat transfer it is of interest to devise methods for evaluating radiation function by using only measurements taken outside the medium.

This paper seeks to determine an unknown radiation function those depend only on the heat in a radiative heat transfer equation.

The plan of this paper is to first formulate a nonlinear IHCP, section 2, the numerical scheme for the solution of (1)-(5) is described in section 3, and finally numerical experiments in section 4, confirm our theoretical results.

2. Description of the problem

In this paper, we consider the problem of determining an unknown function $P(u)$ which is defined on [0,1], and a function $u(x,t)$ satisfying

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad 0 < t < T, \tag{1}$$

$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{2}$$

$$u(0,t) = g(t), \qquad 0 < t < T, \tag{3}$$

$$u_x(1,t) - P(u(1,t)) = \chi(t), \quad 0 < t < T, \tag{4}$$

and the overspecified condition

$$u(1,t) = \phi(t), \quad 0 < t < T, \tag{5}$$

where T is a given positive constant and $f(x)$, $g(t)$, $\chi(t)$ and $\phi(t)$ are piecewise-continuous functions on their domains. The equation (1) may be used to describe the flow of heat in a rod. Hence, we might think of this problem as the problem of determining the unknown radiation term in a rod. The direct problem (1)-(4) has a unique solution [2]. If the function $P(u)$ is given, then there may be no solution for problem (1)-(5). For an unknown $P(u)$, we must therefore provide additional information namely (5) to provide a unique solution $(u, P(u))$ to the inverse problem (1)-(5) [3]. The nonlinear inverse problem (1)-(5) have been previously treated by many authors [1-6].

Theorem 1. For any piecewise-continuous functions f, g, χ, and ϕ there is a unique solution pair (u, P), for the inverse problem (1)-(5).

Proof. [6].

3. Overview of the method

The application of the present numerical method to find the solution of problem (1)-(5) can be described as follows.

First, for linearized nonlinear term in equation (4) we used Taylor's series expansion.

Let $\Psi(\xi_1,...,\xi_n)$ be an infinite differentiable nonlinear function of $\xi_1,...,\xi_n$ then its Taylor s' series expansion is given as

$$\Psi\left(\xi_1,\ldots,\xi_n\right) = \Psi\left(\overline{\xi}_1,\ldots,\overline{\xi}_n\right) +$$

$$\sum_{\lambda=1}^{n}\frac{\partial\Psi}{\partial\xi_\lambda}(\bar{\xi}_1,...,\bar{\xi}_n)(\xi_\lambda-\bar{\xi}_\lambda)+O((\xi_\lambda-\bar{\xi}_\lambda)^2), \quad (6)$$

where the overbar denotes the previously iterated solution.

The function $P(u)$ in equation (4) can be linearized by (6), as follows

$$P(u)=P(\bar{u})+(\frac{\partial}{\partial u}P(u))_{u=\bar{u}}(u-\bar{u})+O((u-\bar{u})^2), \quad (7)$$

where $\bar{u}=\left(\bar{u}_0,\ \bar{u}_1,\ .\ .\ .,\ \bar{u}_N\right)$ denotes the previously iterated solution.

The discretized forms of problem obtained by using the finite difference approximation is given as

$$-ru_{\mu-1,\nu+1}+(1+2r)u_{\mu,\nu+1}-ru_{\mu+1,\nu+1}=u_{\mu,\nu} \quad (8)$$

For $\mu=1$ and $\nu=0$.

$$u_{\mu,0}=f\left(\mu h\right),\quad \nu=0, \quad (9)$$

$$u_{0,\nu}=g\left(\nu k\right),\quad \mu=0, \quad (10)$$

$$(\frac{u_{N+1,\nu}-u_{N-1,\nu}}{2h})-P(\bar{u}_N)-(\frac{\partial P}{\partial u})_{u=\bar{u}_N}(u_{N,\nu}-\bar{u}_N)=\chi(\nu k),\mu=N, \quad (11)$$

where $x=\mu h$, $t=\nu k$, $Nh=1$ and $r=\frac{k}{h^2}$. These equations in matrix form are

$$\Lambda U\quad=\quad\Theta, \quad (12)$$

where

$$\Lambda=\begin{pmatrix}1+2r & -r & 0 & 0\\ -r & 1+2r & -r & 0\\ \vdots & \ddots & \vdots & \\ 0 & -r & 1+2r & -r\\ 0 & 0 & -2r & 1+2r-2rh\frac{\partial P}{\partial u}(\bar{u}_N)\end{pmatrix},$$

and

$$U^t=\left(u_{1,\nu+1}\quad u_{2,\nu+1}\quad .\ .\ .\quad u_{N-1,\nu+1}\quad u_{N,\nu+1}\right),$$

$$\Theta^t=\left(u_{1,\nu}+ru_{0,\nu+1}\quad u_{2,\nu}\quad .\ .\ .\quad u_{N-1,\nu}\quad u_{N,\nu}+\Omega\right),$$

where

$$\Omega=2rhP(\bar{u}_N)-2rh\bar{u}_N\frac{\partial P}{\partial u}(\bar{u}_N)+2rh\chi(\nu k+k).$$

Note that equation (12) is a linear equation.

Theorem 2. If A be a $N\times N$ matrix, $N\geq 2$ and $r\leq\frac{1}{|2-h\frac{\partial P}{\partial u}(\bar{u}_N)|}$ then the finite difference scheme (12) is stable.

Proof. From equation (12) obtains

$$U=\Lambda^{-1}\Theta. \quad (13)$$

The matrix determining the propagation of the error in system (24) is Λ^{-1}, therefore difference scheme (12) will be stable when the modulus of every eigenvalue of Λ^{-1} does not exceed one, that is, when

$$|\frac{1}{\lambda}|\leq 1,$$

where λ is an eigenvalues of Λ.

Application of Bruer's theorem to this matrix, with $a_{ss}=1+2r$ and $P_s=r$ shows that its eigenvalues λ lie on

$$1+r\leq\lambda\leq 1+3r,$$

similarly for rows 2 we obtain

$$1\leq\lambda\leq 1+4r.$$

For the last row λ within the circle,

$$|\lambda-(1+2r-2rh\frac{\partial P}{\partial u}(\bar{u}_N))|\leq 2r,$$

and for stability,

$$|\lambda|\geq 1.$$

Hence

$$r\leq\frac{1}{|2-h\frac{\partial P}{\partial u}(\bar{u}_N)|}.$$

For overall stability, $r\leq\frac{1}{|2-h\frac{\partial P}{\partial u}(\bar{u}_N)|}$. Therefore the modulus of every eigenvalue of A^{-1} less than one.

The LU-Decomposition algorithm is used to solve

$$U^t=\left(u_{1,\nu+1}\quad u_{2,\nu+1}\quad .\ .\ .\quad u_{N,\nu+1}\right).$$

These updated values of U are used to calculate Λ and Θ for iteration. This computational procedure is performed repeatedly until desired convergence is achieved. In this work the polynomial form proposed for the unknown $P(u)$ before performing the inverse calculation. Therefore $P(u)$ approximated as

$$P(u)=a_0+a_1u+a_2u^2+...+a_qu^q, \quad (14)$$

where $\{a_0,a_1,...,a_q\}$ are constants which remain to be determined simultaneously.

To minimize the sum of the squares of the deviations between $u_{N,\nu+1}$ (calculated) and $\phi(\nu(k+1))$, we use least-squares method. The error in the estimate

$$E(a_0,a_1,...,a_q)=\sum_{\nu=0}^{N}(u_{N,\nu+1}-\phi((\nu+1)k))^2, \quad (15)$$

which remain to be minimized. The estimated values of a_i are determined until the value of $E(a_0,a_1,...,a_q)$ is minimum. The computational procedure for estimating unknown coefficients a_i are described as follows:

Step 1. The initial guesses of $a_i > 0$ can be arbitrarily chosen. Therefore, calculated $u_{N,\nu+1}$ can be determined from equation (12). Now we define

$$e_\nu = u_{N,\nu+1} - \phi((\nu+1)k), \quad \nu = 0,1,...,N, \qquad (16)$$

as difference between $u_{N,\nu+1}$ and $\phi((\nu+1)k)$ and so minimizes

$$E(a_0, a_1, ..., a_q) = \sum_{\nu=0}^{N}(e_\nu)^2. \qquad (17)$$

Step 2. To modify a_i the new calculated $u^*_{N,\nu+1}$ can be obtain by expanding in a first order Taylor s' series as

$$u^*_{N,\nu+1} = u_{N,\nu+1} + \sum_{i=0}^{q}\left(\frac{\partial u_{N,\nu+1}}{\partial a_i}\right)h_i, \qquad (18)$$

where $u_{N,\nu+1} = u_{N,\nu+1}(a_0,...,a_q)$ and $u^*_{N,\nu+1} = u_{N,\nu+1}(a_0+h_0,...,a_q+h_q)$. In equation (17), $a_i + h_i = a^*_i$ where h_i denotes the correction for initial values of a_i. Accordingly, the new calculated $u^*_{N,\nu+1}$ with respect to a^*_i can be determined from equation (12). Now define

$$e^*_\nu = u^*_{N,\nu+1} - \phi\left((\nu+1)k\right), \ \nu = 0,1,...,N. \qquad (19)$$

For determining $\dfrac{\partial u_{N,\nu+1}}{\partial a_i}$ we use finite difference representation as follows

$$\Upsilon^i_\nu = \frac{\partial u_{N,\nu+1}}{\partial a_i}$$

$$= \frac{u_{N,\nu+1}(a_0,...,a_i+\tau_i,...,a_q) - u_{N,\nu+1}(a_0,...,a_i,...,a_q)}{\tau_i}, \qquad (20)$$

where i=0,...,q. By substitution (19) in (17) yields

$$u^*_{N,\nu+1} = u_{N,\nu+1} + \sum_{i=0}^{q}(\Upsilon^i_\nu)h_i. \qquad (21)$$

Now from (15), (18) and (20) we find

$$e^*_\nu = e_\nu + \sum_{i=0}^{q}(\Upsilon^i_\nu)h_i. \qquad (22)$$

As shown in equation (16) the error in the estimates can be expressed as

$$E(a_0, a_1, ..., a_q) = \sum_{\nu=0}^{N}(e^*_\nu)^2. \qquad (23)$$

To obtain the minimum value of E with respect to a_i, differentiation of E with respect to the correction h_i will be performed. Thus the correction system corresponding to the values of a_i can be expressed as

$$\sum_{\nu=0}^{N}\left(\sum_{i=0}^{q}\Upsilon^k_\nu \Upsilon^i_\nu h_i\right) = -\sum_{\nu=0}^{N} e_\nu \Upsilon^k_\nu, \ k = 0,1,...,q. \qquad (24)$$

The system (24) may be written in the following matrix form

$$\Lambda\Theta = C, \qquad (25)$$

where

$$\Lambda = \begin{pmatrix} \sum_{\nu=0}^{N}\left(\gamma^0_\nu\right)^2 & \sum_{\nu=0}^{N}\gamma^0_\nu\gamma^1_j & \cdots & \sum_{\nu=0}^{N}\gamma^0_\nu\gamma^q_\nu \\ \sum_{\nu=0}^{N}\gamma^0_\nu\gamma^1_\nu & \sum_{\nu=0}^{N}\left(\gamma^1_\nu\right)^2 & \cdots & \sum_{\nu=0}^{N}\gamma^1_\nu\gamma^q_\nu \\ & & \cdots & \\ \sum_{\nu=0}^{N}\gamma^0_\nu\gamma^q_\nu & \sum_{\nu=0}^{N}\gamma^1_\nu\gamma^q_\nu & \cdots & \sum_{\nu=0}^{N}\left(\gamma^q_\nu\right)^2 \end{pmatrix},$$

$$C = \left(-\sum_{\nu=0}^{N}\gamma^0_\nu e_\nu \ \cdots \ -\sum_{\nu=0}^{N}\gamma^q_\nu e_\nu\right)^T,$$

$$\Theta = \begin{pmatrix} h_0 & h_1 & \cdots & h_q \end{pmatrix}^T.$$

The Tikhonov regularized solution ([11]-[12]-[13]) to the system of linear algebraic equation

$$\Lambda\Theta = C,$$

is given by

$$\Theta_\alpha : \phi_\alpha(\Theta_\alpha) = \min_{\Theta}\phi_\alpha(\Theta),$$

where ϕ_α represents the zeroth order Tikhonov functional given by

$$\phi_\alpha(\Theta) = P\Lambda\Theta - CP^2 + \alpha^2 P\Theta P^2.$$

Solving $\nabla\phi_\alpha(\Theta) = 0$ with respect to Θ, then we obtain, the Tikhonov regularized solution of the regularized equation

$$\Theta_\alpha = (\Lambda^T\Lambda + \alpha^2 I)^{-1}\Lambda^T C.$$

In our computation we use the L-curve scheme to determine a suitable value of α ([11]-[13]).

4. Numerical results and discussion

In this section we are going to demonstrate numerically, some of the results for unknown radiation term in the inverse problem (1)-(5). All the computations are performed on the PC. However, to further demonstrate the accuracy and efficiency of this method, the present problem is investigated and some examples are illustrated.

Example 1. In this example let us consider the following inverse problem

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \qquad (26)$$

$$u(x,0) = cos(x), \quad 0 < x < 1, \qquad (27)$$

$$u(0,t) = exp(-t) \quad 0 < t < T, \qquad (28)$$

$$u_x(1,t) - P(u(1,t)) =$$
$$-1 - (cos(1) + sin(1))exp(-t), 0 < t < T, \qquad (29)$$

with the overspecified condition

$$u(1,t) = cos(1)exp(-t), \qquad 0 < t < T. \qquad (30)$$

The exact solution of this problem is

$$(u(x,t), P(u)) = (cos(x)exp(-t), 1+u),$$

To solve the problem (25)-(29) by the present numerical method, the unknown function $P(u)$ defined as the following form

$$P(u) = a_0 + a_1 u,$$

and the computational procedure for estimating unknown coefficients a_i are repeated until

$$\sum_{\nu=0}^{N} (e_\nu^*)^2 < 10^{-6}.$$

Tables 1 and 2, respectively, shown the values of U in $x = \mu h$ and $t = \nu k$ when $k = \frac{1}{10}$, $h = \frac{1}{6}$. The initial guess of $\{a_0, a_1\}$ is $\{0.6, 0.6\}$ and $\bar{u} = (\bar{u}_\mu)$ is 0.6. The estimated values of a_0 and a_1 are $a_0 = 0.988424$ and $a_1 = 1$.

ν	Numerical $u_{1,\nu}$	Exact $u_{1,\nu}$	Numerical $u_{2,\nu}$	Exact $u_{2,\nu}$	Numerical $u_{3,\nu}$	Exact $u_{3,\nu}$
1	0.893837	0.892299	0.857196	0.855032	0.796178	0.794070
2	0.809388	0.807386	0.776588	0.773665	0.721396	0.718504
3	0.732633	0.730553	0.703126	0.700041	0.653213	0.650129
4	0.663026	0.661032	0.636397	0.633424	0.591233	0.588261
5	0.599954	0.598126	0.575858	0.573145	0.534946	0.532281

Table 1.

ν	Numerical $u_{4,\nu}$	Exact $u_{4,\nu}$	Numerical $u_{5,\nu}$	Exact $u_{5,\nu}$	Numerical $u_{6,\nu}$	Exact $u_{6,\nu}$
1	0.712547	0.711100	0.608543	0.608424	0.486798	0.488886
2	0.645431	0.643430	0.550824	0.550525	0.440183	0.442362
3	0.584361	0.582200	0.498544	0.498135	0.398206	0.400266
4	0.528852	0.526796	0.451051	0.450731	0.360058	0.362175
5	0.478398	0.476665	0.407835	0.407839	0.325268	0.327710

Table 2.

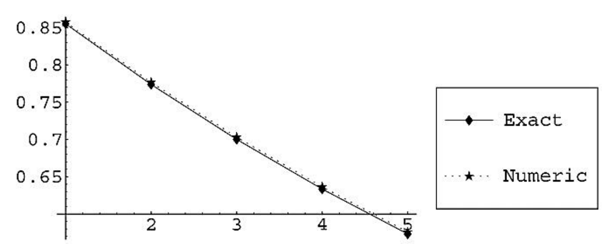

Figure 1. shows the comparison of the surface temperature distributions $u_{\mu,\nu}$ between the exact results and the present numerical results where $\mu = 2$ and $\nu = 1$.

Example 2. In this example let us consider the following inverse problem

$$u_t = u_{xx}, \quad 0 < x < 1,\ t > 0, \qquad (31)$$

$$u(x,0) = sin(x), \quad 0 < x < 1, \qquad (32)$$

$$u(0,t) = 0 \quad 0 < t < T, \qquad (33)$$

$$u_x(1,t) - P(u(1,t)) = [-1 + exp(-t)cos(1) - exp(-2t)sin^2(1)], \quad 0 < t < T, \qquad (34)$$

with the overspecified condition

$$u(1,t) = sin(1)exp(-t), \quad 0 < t < T. \qquad (35)$$

The exact solution of this problem is

$$(u(x,t), P(u)) = (sin(x)exp(-t), 1 + u^2),$$

To solve the problem (30)-(34) by the present numerical method, the unknown function $P(u)$ defined as the following form

$$P(u) = a_0 + a_1 u^2,$$

and the computational procedure for estimating unknown coefficients a_i are repeated until

$$\sum_{\nu=0}^{N} (e_\nu^*)^2 < 10^{-4}.$$

Tables 3 and 4, respectively, shown the values of U in $x = \mu h$ and $t = \nu k$ when $k = \frac{1}{20}$, $h = \frac{1}{6}$. The initial guess of $\{a_0, a_1\}$ is $\{0.6, 0.6\}$ and $\bar{u} = (\bar{u}_\mu)$ is 0.6. The estimated values of a_0 and a_1 are $a_0 = 1.022420$ and $a_1 = 0.999864$.

ν	Numerical $u_{1,\nu}$	Exact $u_{1,\nu}$	Numerical $u_{2,\nu}$	Exact $u_{2,\nu}$	Numerical $u_{3,\nu}$	Exact $u_{3,\nu}$
1	0.157721	0.157805	0.310900	0.311237	0.455026	0.456044
2	0.149674	0.150109	0.294877	0.296058	0.431178	0.433802
3	0.141940	0.142788	0.279584	0.281619	0.408731	0.412645
4	0.134675	0.135825	0.265315	0.267884	0.388028	0.392520
5	0.127956	0.129200	0.252178	0.254819	0.369102	0.373377
6	0.121791	0.122899	0.240158	0.242392	0.351846	0.355167

Table 3.

ν	Numerical $u_{4,\nu}$	Exact $u_{4,\nu}$	Numerical $u_{5,\nu}$	Exact $u_{5,\nu}$	Numerical $u_{6,\nu}$	Exact $u_{6,\nu}$
1	0.585597	0.588215	0.697960	0.704078	0.786871	0.800432
2	0.554230	0.559524	0.659857	0.669740	0.744314	0.761394
3	0.525408	0.532236	0.626074	0.637076	0.707970	0.724261
4	0.499238	0.506278	0.595910	0.606006	0.675824	0.688938
5	0.575512	0.481587	0.568740	0.576450	0.646875	0.655338
6	0.453949	0.458100	0.544071	0.548336	0.620488	0.623377

Table 4.

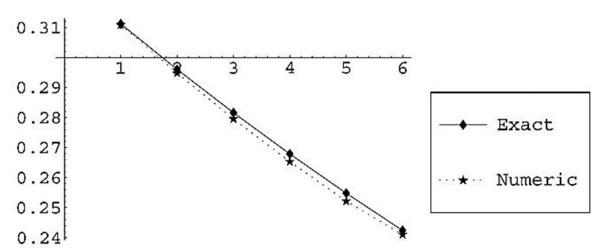

Figure. 2 shows the comparison of the surface temperature distributions $u_{\mu,\nu}$ between the exact results and the present numerical results where $\mu = 2$ and $\nu = 1$.

5. Conclusion

A numerical method to estimate unknown radiation term is proposed for an inverse problem of linear heat equation with nonlinear boundary condition and the following results are obtained.

1. The present study, successfully applies the numerical method involving the finite difference method in conjunction with the least-squares scheme to a non-linear problem.
2. From the illustrated examples it can be seen that the proposed numerical method is efficient and accurate to estimate the unknown radiation term.
3. The present method has merit in that it is independent of initial guesses of $\{a_0, a_1, ..., a_t\}$ and $\bar{u} = (\bar{u}_i)$. We also apply other different sets of the initial guesses, such as $\{\bar{u}_0, \bar{u}_1, ..., \bar{u}_N\} = \{0.3, ..., 0.3\}$, $\{0.6, ..., 0.6\}$ and $\{1.2, ..., 1.2\}$, results show that the effect of the initial guesses on the accuracy of the estimates is not significant for the present method.

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