Retractions of Spatially Curved Robertson-Walker Space

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Abstract: Our aim in the present paper is to introduce and study new types of retractions of open flat Robertson –Walker W^4model. Types of the deformation retracts of open flat Robertson –Walker W^4model are obtained. The relations between the folding and the deformation retract are deduced. Types of minimal retractions are presented. New types of homotopy maps are deduced. New types of conditional folding are presented. Some commutative diagrams are obtained.

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Keywords. Retraction ; Deformation retract ; Folding ; Open flat Robertson –Walker W^4model**.**

1. Introduction

Flat Robertson –Walker space represents one of the most intriguing and emblematic discoveries in the history of geometry. Although if it were introduced for a purely geometrical purpose, they came into prominence in many branches of mathematics and physics. This association with applied science and geometry generated synergistic effect: applied science gave relevance to flat Robertson –Walker space and flat Robertson –Walker space allowed formalizing practical problems[1, 10, 11, 14, 15, 17]. Most folding problems are attractive from a pure mathematical standpoint, for the beauty of the problems themselves. The folding problems have close connections to important industrial applications Linkage folding has applications in robotics and hydraulic tube bending. Paper folding has application in sheet-metal bending, packaging, and air–bag folding. Also, used folding to solve difficult problems related to shell structures in civil engineering and aero space design, namely buckling instability. Isometric folding between two Riemannian manifold may be characterized as maps that send piecewise geodesic segments to a piecewise geodesic segments of the same length. For a topological folding the maps do not preserves lengths, i. e. A map$\Im : M \to N$, where M and N are C^{∞}–Riemannian manifolds of dimension m, n respectively is said to be an isometric folding of M into$N$, iff for any piecewise geodesic path $\Upsilon : J \to M$,the induced path $\Im \circ \Upsilon : J \to N$ is a piecewise geodesic and of the same length as. If $\Im$ does not preserve length, then $\Im$ is a topological folding [2, 3, 6, 8, 9, 12, 13].

A subset A of a topological space X is called a retract of X if there exists a continuous map $r : X \to A$ such that $r(a) = a \ \forall \ a \in A$, where A is closed and X is open [2, 3, 4, 5, 6, 7] Also, a subset A of a topological space X is a deformation retracts of X if there exists a retraction $r : X \to A$ and a homotopy $\emptyset$: $X \times I \to X$ such that:

$$\left.\begin{aligned} &\emptyset\,(x,0) = x \\ &\emptyset\,(x,1) = r(x) \\ &\emptyset\,(a,t) = a,\ a \in A,\ t \end{aligned}\right\} \begin{aligned} &x \in X \\ & \\ &\in [0,1] \end{aligned} \quad [9, 12, 13, 16]$$

The flat Robertson –Walker W^4line element $ds^2 = -dt^2 - a^2\,(t)\,(\,dX^2 - dY^2 - dZ^2)$ is one example of a homogeneous isotropic cosmological spacetime geometry, but not the only one. the general Robertson –Walker W^4 line element for a homogeneous isotropic universe has the form $ds^2 = -dt^2 - a^2\,(t)\ d\ell^{\,2}$, where $d\ell^{\,2}$ is the line element of a homogeneous, isotropic three - dimensional space . There are only three possibility for this. Let's now look at the open flat Robertson –Walker W^4model. In the present work we give first some rigorous definitions of retractions, folding and deformation retraction as well as important theorems of open flat Robertson –Walker W^4model[10, 11, 14, 15, 17].

2. Main results

Theorem1. The retractions of the open flat Robertson –Walker W^4 model are unit hyperboloid, hyperbolic, hypersphere, circle and minimal manifolds .

Proof .Consider the open flat Robertson -Walker W^4 model with metric

$d\ell^2 = d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\emptyset^2)$ (1)

The coordinate of the open flat Robertson -Walker W^4 are

$x_1 = \sinh\chi \sin\theta \cos\emptyset$, $x_3 = \sinh\chi\cos\theta$,

$x_2 = \sinh\chi \sin\theta \sin\emptyset$, $x_4 = \cosh\chi$ (2)

where the ranges are, $0 \le \theta < \pi$, $0 \le \emptyset < 2\pi$ and $0 \le \chi < \infty$

Now, we use Lagrangian equations

$\frac{d}{ds}\left(\frac{\partial T}{\partial \dot\varphi_i}\right) - \frac{\partial T}{\partial \varphi_i} = 0$, $i = 1, 2, 3.$

to find a geodesics which is a subset of the open flat Robertson -Walker space W^4. Since $T = \frac{1}{2}\{\dot\chi^2 - \sinh^2\chi(\dot\theta^2 - \sin^2\theta\,\dot\emptyset^2)\}$.

Then the Lagrangian equations for open flat Robertson -Walker space W^4 are

$\frac{d}{ds}(\dot\chi) - (\sinh\chi\cosh\chi(\dot\theta^2 - \sin^2\theta\,\dot\emptyset^2)) = 0$. (3)

$\frac{d}{ds}(\sinh^2\chi\,\dot\theta) - (\sinh^2\chi\,\sin\theta\cos\theta\,\dot\emptyset^2) = 0$. (4)

$\frac{d}{ds}(\sinh^2\chi\sin^2\theta\dot\emptyset) = 0$. (5)

From equation (5) we obtain $\sinh^2\chi \sin^2\theta\,\dot\emptyset =$ constant say β_1, if $\beta_1 = 0$, we obtain the following cases:

If initially θ equal $\frac{\pi}{6}$ or $\frac{\pi}{4}$ and $\frac{\pi}{3}$ hence we obtain the following geodesics unit hyperboloid H_1^3, $H_2^3$and H_3^3 respectively. Also, if $\theta = \frac{\pi}{2}$, hence we obtain the following coordinate of open flat Robertson -Walker space W^4 given by

$x_1 = \sinh\chi\cos\emptyset$, $x_2 = \sinh\chi\sin\emptyset$, $x_3 = 0$, $x_4 = \cosh\chi$.

which is a hyperbolic H_1^2, $-x_4{}^2 + x_1{}^2 + x_2{}^2 + x_3{}^2 = -1$, which is a geodesic and retraction. Now, If $\emptyset = \frac{\pi}{6}$ or $\frac{\pi}{4}$ and $\frac{\pi}{3}$ hence we get the unit hyperboloid retractions H_4^3, H_5^3 and H_6^3 in open flat Robertson -Walker space W^4 respectively. Also, in a special case if $\emptyset = \frac{\pi}{2}$ or π and $\frac{3\pi}{2}$ hence we get the hyperbolic geodesics retraction H_2^2, H_3^2 and H_4^2 in open flat Robertson -Walker space W^4 respectively. In a special case if $\chi = \frac{\pi}{2}$, hence we get the coordinate of open flat Robertson -Walker space W^4 which represented by

$x_1 = \sin\theta\cos\emptyset$, $x_2 = \sin\theta\sin\emptyset$, $x_3 = \cos\theta$, $x_4 = 0$.

which is sphere S_1^2, $-x_4{}^2 + x_1{}^2 + x_2{}^2 + x_3{}^2 = 1$, which is a geodesic retraction. Also, If $\emptyset = 90$, and $\chi = 90$ we obtain the retraction, $S^1 = (0, \sin\theta, \cos\theta, 0)$, which is a circle S^1. Again, If $\chi = \pi$, we get the following minimal geodesic $W^0(0, 0, 0, 1)$ in open flat Robertson -Walker space W^4.

In what follows, we present some cases of the deformation retract of open flat Robertson -Walker space W^4. The deformation retract of open flat Robertson -Walker space W^4 is $\eta : W^4 \times I \to W^4$, Where I is the closed interval [0,1], be present as

$\eta(x,h)$: $(\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, \sinh\chi\cos\theta, \cosh\chi) \times I \to (\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, \sinh\chi\cos\theta, \cosh\chi)$

The deformation retract of open flat Robertson -Walker W^4 space into the minimal geodesic W^0 is

$\eta(m,h) = (1 + h)\{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, \sinh\chi\cos\theta, \cosh\chi\} + \tan\frac{\pi h}{4}\{0, 0, 0, 1\}$. where

$\eta(m,0) = \{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, \sinh\chi\cos\theta, \cosh\chi\}$

and $\eta(m,h)$ $\{0, 0, 0, 1\}$

The deformation retract of open flat Robertson -Walker W^4 space into the hyperboloid H_1^2 is

$\eta(m,h) = \cos\frac{\pi h}{2}\{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, \sinh\chi\cos\theta, \cosh\chi\} + \sin\frac{\pi h}{2}\{\sinh\chi\cos\emptyset, \sinh\chi\sin\emptyset, 0, \cosh\chi\}$

Now, we are going to discuss the folding $\mathfrak{F}$ of the open flat Robertson -Walker space W^4. Let $\mathfrak{F} : W^4 \to W^4$, where

$\mathfrak{F}(x_1, x_2, x_3, x_4) = (x_1, x_2, |x_3|, x_4)$. (6)

An isometric folding of open flat Robertson -Walker space W^4 into itself may be defined by

$\mathfrak{F} : \{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, \sinh\chi\cos\theta, \cosh\chi\} \to \{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, |\sinh\chi\cos\theta|, \cosh\chi\}$

The deformation retract of the folded open flat Robertson -Walker space $\mathfrak{F}(W^4)$ into the folded geodesic $\mathfrak{F}(W^0)$ is

$\eta_{\mathfrak{F}} : \{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, |\sinh\chi\cos\theta|, \cosh\chi\} \times I \to \{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, |\sinh\chi\cos\theta|, \cosh\chi\}$

with

$\eta_{\mathfrak{F}}(m,h) = (1 + h)\{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, |\sinh\chi\cos\theta|, \cosh\chi\} + \tan\frac{\pi h}{4}\{0, 0, 0, 1\}$.

The deformation retract of the folded open flat Robertson -Walker space $\mathfrak{F}(W^4)$ into the folded geodesic $\mathfrak{F}(H_1^2)$ is

$\eta_{\mathfrak{F}}(m,h) = \cos\frac{\pi h}{2}\{\sinh\chi\sin\theta\cos\emptyset, \sinh\chi\sin\theta\sin\emptyset, |\sinh\chi\cos\theta|, \cosh\chi\} + \sin\frac{\pi h}{2}\{\sinh\chi\cos\emptyset, \sinh\chi\sin\emptyset, 0, \cosh\chi\}$.

Then, the following theorem has been proved.

Theorem 2. Under the defined folding and any folding homeomorphic to this type of folding , the deformation retract of the folded open flat Robertson -Walker space $\mathfrak{F}$ (W^4) into the folded geodesics is the same as the deformation retract of open flat Robertson -Walker space W^4 into the geodesics.

Now, let the folding be defined by $\mathfrak{F}^* : W^4 \rightarrow W^4$, where

$\mathfrak{F}^*$ (x_1 , x_2 , x_3 , x_4) = ($|x_1|$, x_2, x_3, x_4) (7)

The isometric folded open flat Robertson -Walker space $\mathfrak{F}^*$ (W^4) is

$\overline{R}$ = { | $\sinh\chi$ $\sin\theta$ $\cos\emptyset$| , $\sinh\chi$ $\sin\theta$ $\sin\emptyset$, $\sinh\chi$ $\cos\theta$,$\cosh\chi$}

The deformation retract of the folded open flat Robertson -Walker space $\mathfrak{F}^*$ (W^4) into the folded geodesic $\mathfrak{F}^*$ (H_2^2) is

$\eta_{\mathfrak{F}^*}(m,h)$ = (1 − h) {| $\sinh\chi$ $\sin\theta$ $\cos\emptyset$| , $\sinh\chi$ $\sin\theta$ $\sin\emptyset$, $\sinh\chi\cos\theta$, $\cosh\chi$ } + $\tan\frac{\pi h}{4}$ { 0 ,$\sinh\chi\sin\theta$, $\sinh\chi$ $\cos\theta$, $\cosh\chi$ } .

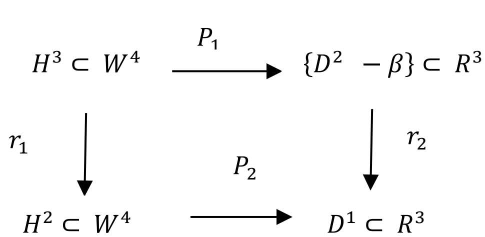
The deformation retract of the folded open flat Robertson -Walker space $\mathfrak{F}^*$ (W^4) into the folded geodesic $\mathfrak{F}^*$ (H_3^2) is

$\eta_{\mathfrak{F}^*}(m,h)$ = $\cos\frac{\pi h}{2}$ { | $\sinh\chi$ $\sin\theta$ $\cos\emptyset$| , $\sinh\chi$ $\sin\theta$ $\sin\emptyset$, $\sinh\chi\cos\theta$, $\cosh\chi$ } + $\sin\frac{\pi h}{2}$ { |$\sinh\chi$ $\sin\theta$| , 0 ,$\sinh\chi\cos\theta$, $\cosh\chi$ }

Then, the following theorem has been proved.

Theorem 3. Under the defined folding and any folding homeomorphic to this type of folding , the deformation retract of the folded open flat Robertson - Walker space $\mathfrak{F}^*$ (W^4) into the folded geodesic is different from the deformation retract of open flat Robertson -Walker space W^4 into the geodesics .

Theorem 4 . Let $H^3 \subset W^4$ be a hyperboloid in open flat Robertson- Walker space which is homeomorphic to $D^2 \subset R^3$, and $r_1 : H^3 \rightarrow H^2$ be a retraction . Then, there is an induced retraction r_2 :{ D^2 − β} → D^1 such that the following diagram is commutative



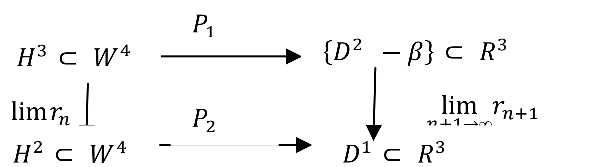
Proof . Since $r_1 : H^3 \rightarrow H^2$ and $r_2 : \{D^2 - \beta\} \rightarrow D^1$ be defined as

r_1 {$\sinh\chi$ $\sin\theta$ $\cos\emptyset$, $\sinh\chi$ $\sin\theta$ $\sin\emptyset$, $\sinh\chi$ $\cos\theta$, $\cosh\chi$} = {$\sinh\chi$ $\cos\emptyset$, $\sinh\chi$ $\sin\emptyset$, 0 , $\cosh\chi$} and

r_2 {$\sin\theta$ $\cos\emptyset$, $\sin\theta$ $\sin\emptyset$, $\cos\theta$, 0 } ={$\cos\emptyset$, $\sin\emptyset$, 0,0 } .

Under the homeomorphism map $P_1 : H^3 \subset W^4 \rightarrow \{D^2 - \beta\} \subset R^3$ and $P_2 : H^2 \subset W^4 \rightarrow D^1 \subset R^3$. This proves that the diagram is commutative .

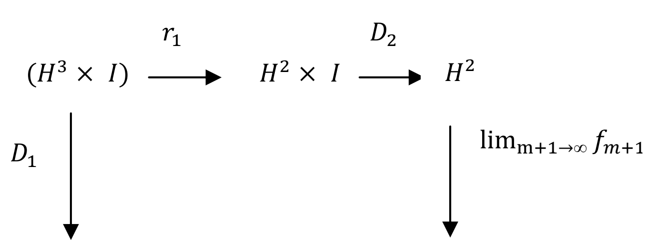
Theorem 5. Let $H^3 \subset W^4$ be a hyperboloid which is homeomorpic to $D^2 \subset R^3$, and $\lim_{n\rightarrow\infty} r_n : H^3 \rightarrow H^2$ be a limit retraction. Then, there is an induced limit retraction $\lim_{n\rightarrow\infty} r_{n+1} : D^2 \rightarrow D^1$ such that the following diagram is commutative

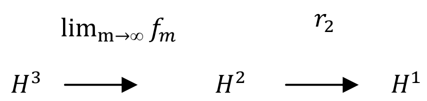


Proof. Since $\lim_{n\rightarrow\infty} r_n : H^3 \rightarrow H^2$ and $\lim_{n+1\rightarrow\infty} r_{n+1}$: $\{D^2 - \beta\} \rightarrow D^1$. Under the homeomorphism map P_1 : $H^3 \subset W^4 \rightarrow \{D^2 - \beta\} \subset R^3$ and $P_2 : H^2 \subset W^4 \rightarrow D^1 \subset R^3$. This proves that the diagram is commutative .

Theorem 6 . If the deformation retract of the hyperboloid $H^3 \subset W^4$ is $D: H^3 \times I \rightarrow H^3$, the retraction of $H^3 \subset W^4$ is $r : H^3 \rightarrow H^2, H^2 \subset H^3$ and the limit of the folding of H^3 is $\lim_{m\rightarrow\infty} f_m : H^3 \rightarrow H^2$. Then there are induces deformations retract, retractions , and the limit of the foldings such that the following diagram is commutative.

Proof . Let the deformation retract of $H^3 \subset W^4$ is $D_1: H^3 \times I \rightarrow H^3$, the retraction of $H^3 \times I$ is defined by r_1: ($H^3 \times I$) → $H^2 \times I$, $\lim_{m\rightarrow\infty} f_m$: D_1 ($H^3 \times I$) → H^2, the deformation retract of r_1 ($H^3 \times I$) is D_2: r_1 ($H^3 \times I$) → H^2, the retraction of $\lim_{m\rightarrow\infty} f_m$ (D_1 (($H^3 \times I$))) is given by r_2 : $\lim_{m\rightarrow\infty} f_m$ (D_1 ($H^3 \times I$)) → H^1, and $\lim_{m+1\rightarrow\infty} f_{m+1}$: D_2 (r_1 ($H^3 \times I$) → H^1, H^1 is a 1-dimensional space . Hence, the following diagram is commutative .

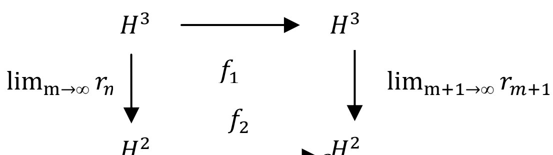




i.e. $\lim_{m+1\to\infty} f_{m+1} \circ D_2 \circ r_1\ (H^3 \times I) = r_2 \circ \lim_{m\to\infty} f_m \circ D_1$

Theorem 7. Let $H^3 \subset W^4$ be the hyperboloid, then the relation between the folding $f : H^3 \to H^3$, and the limit of the retractions $\lim_{m\to\infty} r_m : H^3 \to H^2$, discussed from the following commutative diagram .

Proof . Let the folding is $f_1 : H^3 \to H^3$, the limit of the retractions of H^3 and $f_1\ (H^3)$ are $\lim_{m\to\infty} r_m : H^3 \to H^2$ and $\lim_{m+1\to\infty} r_{m+1} : f_1\ (H^3) \to H^2$, and $f_2\ : (\lim_{m\to\infty} r_m\ (H^3)) \to H^2$. Then, the following commutative diagram.



i.e. $\lim_{m+1\to\infty} r_{m+1} \circ f_1\ (H^3) = f_2 \circ \lim_{m\to\infty} r_m\ (H^3)$.

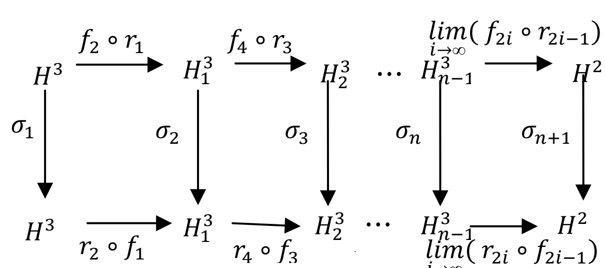
Theorem 8 . Let the retraction of H^3 is $r : H^3 \to H^2$, $H^2 \subset H^3$, and the folding of H^3 is $f : H^3 \to H^3$, then

(i) - $f_2 \circ r_1(H^3) = r_2 \circ f_1\ (H^3)$

(ii)- $\sigma_{n+1} \circ (\lim_{i\to\infty}(f_{2i} \circ r_{2i-1})\ (\cdots (f_4 \circ r_3\ (f_2 \circ r_1\ (H^3)))\ \cdots)) = (\lim_{i\to\infty}(r_{2i} \circ f_{2i-1})(\cdots (r_4 \circ f_3\ (r_2 \circ f_1\ (H^3)))\ \cdots)) \circ \sigma_1$.

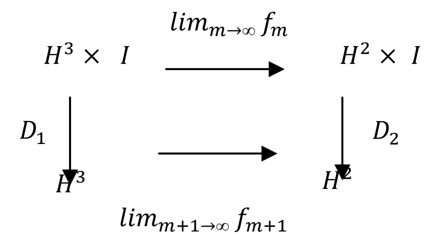
Proof . (i) **-** Let the retraction of the hyperboloid in open Robertson- Walker space $H^3 \subset W^4$ is $r_1 : H^3 \to H^2$, $f_1 : H^3 \to H^3$, the retraction of $f_1\ (H^3)$ is $r_2 : f_1(H^3) \to H^2$, and the folding of $r_1\ (H^3)$ is $f_2 : r_1\ (H^3) \to H^2$. Then $f_2 \circ r_1(H^3) = r_2 \circ f_1\ (H^3)$.

(ii) **-** Let $f_{2i} \circ r_{2i-1}$ and $r_{2i} \circ f_{2i-1}$ are the compositions between the retractions and the foldings of H^3into itself .Also, σ_i are the homeomorphisms .Then



Theorem 9. Given the deformation retract of $H^3 \subset W^4$ is $D: H^3 \times I \to H^3$, the limit of the folding of $H^3 \times I$ is $\lim_{m\to\infty} f_m : H^3 \times I \to H^2 \times I$ .Then, the following diagram is commutative.

Proof . Let the limit of the folding of $(H^3 \times I)$ is $\lim_{m\to\infty} f_m : H^3 \times I \to H^2 \times I$, the deformation retract of $H^3 \subset W^4$ is $D_1: H^3 \times I \to H^3$, the limit of the folding of $D_1\ (H^3 \times I)$ is $\lim_{m+1\to\infty} f_{m+1} : D_1\ (H^3 \times I) \to H^2$, and the deformation retract of $\lim_{m\to\infty} f_m\ (H^3 \times I)$ is $D_2 : \lim_{m\to\infty} f_m\ (H^3 \times I) \to H^2$. Hence



i.e. $D_2 \circ \lim_{m\to\infty} f_m\ (H^3 \times I) = \lim_{m+1\to\infty} f_{m+1} \circ D_1\ (H^3 \times I)$.

Theorem 10. The composition of strong deformation retract of the hyperboloid $H^3 \subset W^4$ is a minimal retraction.

Proof . Now consider the following continuous map $\eta : H^3 \times [0,1] \to H^3$, such that $\eta(x, s) = \beta(x, \frac{s}{1-s})$, then it is easy to see that

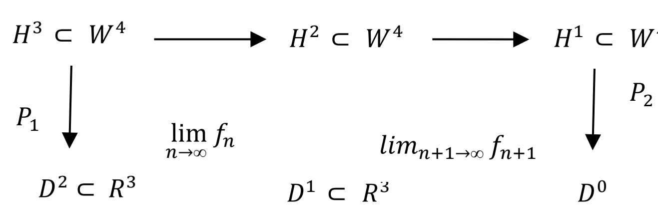
$$\eta(x, 0) = \beta(x,\ 0) = 0\ ,$$
$$\eta(x, 1) = \lim_{s\to 1} \beta(x, \frac{s}{1-s}) = S_1^2 \subset H^3\ ,$$
$$\eta(y, s) = \beta(y, \frac{s}{1-s}) = S^1 \subset S_1^2.$$

The deformation retract of the circle $S^1 \subset S_1^2$ onto minimal retraction (0,1) is given in polar coordinates by

$$\mu(r\, e^{i\theta}) = \begin{cases} r e^{i(1-r)\theta}, |\theta| \le \frac{\pi}{2}\ , \\ r e^{i\{\theta-(\pi-\theta)r\}}, \frac{\pi}{2} \le \theta \le \pi\ , \\ r e^{i\{\theta+(\pi+\theta)r\}}, -\pi \le \theta \le -\frac{\pi}{2} \end{cases}$$

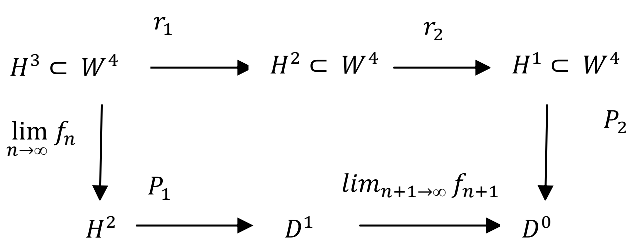
i.e. $\mu \circ \eta$ is a minimal retraction.

Theorem 11. Let $H^3 \subset W^4$ be a hyperboloid in open Robertson - Walker space which is homeomorphism to $D^2 \subset R^3$, $P_1 : H^3 \to D^2$, the retraction $r_1 : H^3 \to H^2$, and the limit folding of D^2 is $\lim_{n\to\infty} f_n : D^2 \to D^1$. Then they are induces retraction, limit folding, and homeomorphism map such that the following diagram is commutative



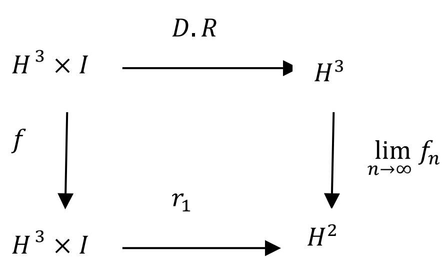
Proof . Let the homeomorphism map , $P_1 : H^3 \to D^2$, and $r_1 : H^3 \to H^2$, also, $\lim_{n\to\infty} f_n : D^2 \to D^1$, the retraction of $r_1 (H^3)$ is $r_2 : H^2 \to H^1$, the limit folding of $\lim_{n\to\infty} f_n (D^2)$ is given by $\lim_{n+1\to\infty} f_{n+1} : D^1 \to D^0$, and $P_2 : r_2 (r_1 (H^3) \to D^0$. This proves that the diagram is commutative.

Theorem 12. If the limit folding of the hyperboloid $H^3 \subset W^4$ is $\lim_{n\to\infty} f_n : H^3 \to H^2$, the retraction of $H^3 \subset W^4$ is $r_1 : H^3 \to H^2$, and the homeomorphism map of $H^2 \subset W^4$ is $P_1 : H^2 \to D^1$. Then they are induces limit retractions, limit folding, and homeomorphism map such that the following diagram is commutative :



Proof . Consider the limit folding of the hyperboloid $H^3 \subset W^4$ is $\lim_{n\to\infty} f_n : H^3 \to H^2$, the retraction of $H^3 \subset W^4$ is $r_1 : H^3 \to H^2$, and the homeomorphism map of $H^2 \subset W^4$ is $P_1 : H^2 \to D^1$, the limit retraction of $r_1 (H^3)$ is $\lim_{m\to\infty} r_m : H^2 \to H^1$, the limit folding of of $P_1 (H^2)$ is $\lim_{n+1\to\infty} f_{n+1} : D^1 \to D^0$, and $P_2 : \lim_{m\to\infty} r_m (r_1 (H^3)) \to D^0$. This proves that the diagram is commutative.

Theorem 13. Let $D.R : H^3 \times I \to H^3$ be a deformation retract of open Robertson- Walker space. and $r_1 : H^3 \times I \to H^2$ be a retraction , also, $f : H^3 \times I \to H^3 \times I$ be a folding . Then is an induced limit folding $\lim_{n\to\infty} f_n : H^3 \to H^2$ such that the following diagram is commutative :



Proof . Let $D.R : H^3 \times I \to H^3$, and the folding $f : H^3 \times I \to H^3 \times I$, also, $r_1 : H^3 \times I \to H^2$, and the limit folding $\lim_{n\to\infty} f_n : H^3 \to H^2$. This proves that the diagram is commutative.

Conclusion

The present paper deals what we consider to be open flat Robertson –Walker W^4model. The retractions of open flat Robertson –Walker W^4model are presented . The deformation retract of open flat Robertson –Walker W^4model will be deduced .The connection between folding and deformation retract is achieved . New types of conditional folding are presented. Also, the relations between the limits of folding and retractions are discussed .Some commutative diagrams are presented.

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