On a Numerical Method for Solving Fredholm - Volterra Integral Equation

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Abstract: In this paper, the existence and uniqueness of solution of Fredholm – Volterra integral equation (**F-VIE**) of the first kind is considered in the space $L_2[-1,1]\times C(0,T), T<1$. Then, a numerical method is used to reduce this type of equation to a system of Fredholm integral equations (**SFIEs**). After this, Toeplitz matrix method (**TMM**) is used to obtain a linear algebraic system (**LAS**). Finally, the linear algebraic system is solved numerically, when the singular kernel takes the logarithmic form and Carleman function. The error, in each case, is calculated.

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1. Introduction

The **F- VIF** of the first kind with singular kernel can be solved analytically, using the following methods: Cauchy method, orthogonal polynomial method, Potential theory method and Krein's method. The importance of **F – VIE** of the first kind and contact problem came from the work of Abdou [1]. Where, the solution of **F – VIE** of the first kind in one, two and three dimensional has been obtained, analytically using separation of variables method. Beside this, the relations between the **F – VIE** and contact problems in the theory of elasticity have been discussed in [2, 4]. The references [5-10] contain some different methods to obtain the solution of the singular integral equations numerically.

Consider the linear **IE**:

$$\phi(x,t)=f(x,t)+\lambda\int_{-1}^{1}k(x,y)\phi(y,t)dy+\lambda\int_{0}^{t}F(t,\tau)\phi(x,\tau)d\tau \quad (1)$$

The linear **IE** (1) is called **F – VIE** of the second kind, in the space $L_2[-1,1]\times C[0,T], T<1$. In Eq. (1), the **FI** term is considered in position and its kernel $k(x,y)$ has a singular term. While the **VI** term is considered in time and its kernel $F(t,\tau)$ is positive and continuous for all $t,\tau\in[0,T], T<1$.

In order to guarantee the existence of a unique solution of Eq. (1), we assume the following conditions:

(i) The kernel $k(x,y)$ satisfies the discontinuity condition:

$$\left[\int_{-1}^{1}\int_{-1}^{1}|k(x,y)|^2\,dydx\right]^{\frac{1}{2}}=c<\infty \quad , \quad (c \text{ is a constant})$$

(ii) The kernel $F(t,\tau)\in C([0,T]\times[0,T])$, $0\le\tau\le t\le T<1$, satisfies :

$|F(t,\tau)|\le M,\ \forall\, t,\tau\in[0,T]$, M is a constant ,

(iii) The given function $f(x,t)$ with its partial derivatives with respect to x *and* t are continuous in $L_2[-1,1]\times C[0,T]$ where ,

$$\|f(x,t)\|=\max_{0\le t\le T}\int_{0}^{t}\left[\int_{-1}^{1}|f(x,\tau)|^2\,dx\right]^{\frac{1}{2}}d\tau=H, \qquad H \text{ is a constant}$$

(iv) The unknown function $\varphi(x,t)$ satisfies Lipschitz condition with respect to position and Hölder condition with respect to time.

In this paper, the **F – VIE** of the second kind is solved numerically in the space $L_2[-1,1]\times C(0,T), T<1$. The kernel of **FI** term is considered in the logarithmic form and Carleman function with respect to position, while the kernel of **VI** term is considered as a positive continuous function in time. In section two, using a numerical method, the **F– VIE** will reduce to linear **SFIEs**. In section three **TMM,** as the best numerical methods, is discussed and applied to solve the **FIE** of the second kind with discontinuous kernel. In section four, numerical results are computed and the estimating error, in each case, is calculated.

2. The System of Fredholm Integral Equations

For representing (1) as **SFIEs**, we divide the interval [0,T] as $0 \le t \le T < 1$, where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_N = T$, then let $t = t_k$, $k = 0,1,2,\cdots,N$. Using the quadrature formula, see [11-13], the **VI** term in (1) becomes

$$\int_0^{t_k} \varphi(x,\tau)F(t_k,\tau)d\tau = \sum_{j=0}^{k} u_j F(t_k,t_j)\varphi(x,t_j) + O\left(\hbar_k^{\tilde{p}+1}\right), \quad \left(\hbar_k \to 0, \tilde{p} > 0\right) \tag{2}$$

Where,

$\hbar_k = \max_{0 \le j \le k} h_j, \quad h_j = t_{j+1} - t_j$

The values of k and the constant $\tilde{p}$ depend on the number of derivatives of the kernel of **VI** term $F(t,\tau)$, for all $\tau \in [0,T]$, with respect to t, for example if the function of time $F(t,\tau) \in C^3[0,T]$, then we have $\tilde{p} = 3$, $\tilde{p} \approx k$, and $u_0 = \frac{1}{2}h_0$, $u_k = \frac{1}{2}h_k$, $u_i = h_i$, $(i \ne 0, k)$. Using Eq. (2) in Eq. (1), after letting $t = t_k$, $k = 1,2,...,N$, we have

$$\phi_k(x) = f_k(x) + \lambda\int_{-1}^{1} k(x,y)\phi_k(y)dy + \lambda\sum_{j=0}^{k} u_j F_{k,j}\phi_j(x) \tag{3}$$

The formula (3) can be adapted in the form

$$\left(1 - \lambda u_k F_{k,k}\right)\phi_k(x) = f_k(x) + \lambda\int_{-1}^{1} k(x,y)\phi_k(y)dy + \lambda\sum_{j=0}^{k-1} u_j F_{k,j}\phi_j(x) \tag{4}$$

Here, we used the following notations:

$\varphi_k(x) = \varphi(x,t_k),\ f_k(x) = f(x,t_k),\ F_{k,j} = F(t_k,t_j)$

The formula (4) represents **SFIEs**. In this aim, we write (4) in the form

$$\mu_n\phi_n(x) = G_n(x) + \lambda\int_{-1}^{1} k(x,y)\phi_n(y)dy \tag{5}$$

where $\mu_n = \left(1 - \lambda u_n F_{n,n}\right)$, and

$$G_n(x) = f_n(x) + \lambda\sum_{j=0}^{n-1} u_j F_{n,j}\varphi_j(x), n = 0,1,2,\cdots,N \cdot$$

The formula (5) leads to say that, we have N unknown functions $\varphi_n(x)$ corresponding to the time interval $[0,T]$, T < 1. In addition, for all values $\mu_n = \text{constant} \ne 0$, we have **SFIEs** of the second kind, while for all values $\mu_n = 0$, we have the integral system of the first kind.

3. The Toeplitz Matrix Method, see [9, 14]

Here, in this section we present **TMM,** as the best numerical method for solving the singular integral equations, to obtain numerically the solution of **FIE** of the second kind with singular kernel. The idea of this method is to obtain a system of $2N+1$ linear algebraic equations, where $2N+1$ is the number of the discretization points used.

In this aim, consider the **FIE** of the second kind:

$$\phi(x) - \lambda\int_{-a}^{a} k(x,y)\,\phi(y)\,dy = f(x) \tag{6}$$

Then, write the integral term of (6) in the form

$$\int_{-a}^{a} k\left(\left|x-y\right|\right)\varphi(y)dy = \sum_{n=-N}^{N-1}\int_{nh}^{nh+h} k\left(\left|x-y\right|\right)\varphi(y)\,dy, \quad \left(h = \frac{2a}{N}\right) \tag{7}$$

Approximate the integral in the right hand side of Eq. (7) by

$$\int_{nh}^{nh+h} k\left(\left|x-y\right|\right)\phi(y)dy = A_n(x)\phi(nh) + B_n(x)\phi(nh+h) + R \tag{8}$$

Where, $A_n(x)$ and $B_n(x)$ are two arbitrary functions will be determined and R is the estimate error. Putting $\varphi(x) = 1, x$ in Eq. (8) yields a set of two equations in terms of the two functions $A_n(x)$ and $B_n(x)$, where in this choosing we have $R = 0$. By solving the results, the functions $A_n(x)$ and $B_n(x)$ will take the forms

$$A_n(x) = \frac{1}{h}\left[(nh+h)\,I(x) - J(x)\right],\ B_n(x) = \frac{1}{h}\left[J(x) - nh\,I(x)\right], \tag{9}$$

The values of $I(x)$ and $J(x)$ are

$$I(x) = \int_{nh}^{nh+h} k\left(\left|x-y\right|\right)dy,\ \ J(x) = \int_{nh}^{nh+h} y\,k\left(\left|x-y\right|\right)dy \quad , \tag{10}$$

Hence, the relation (7), becomes

$$\int_{-a}^{a} k\left(\left|x-y\right|\right)\phi(y)\,dy = \sum_{n=-N}^{N} D_n(x)\,\phi(nh) \tag{11}$$

Where

$$D_n(x) = \begin{cases} A_{-N}(x), & n = -N \\ A_n(x) + B_{n-1}(x), & -N < n < N \\ B_{N-1}(x), & n = N \end{cases}$$

The integral equation (6), after putting $x = mh$, becomes

$$\phi(mh) - \lambda\,a_{n,m}\,\phi(nh) = f(mh) \tag{12}$$

The function ϕ represents a vector consists of 2N+1 element. While, $a_{n,m}$ is a matrix whose elements are given by

$$a_{n,m} = a'_{n,m} + g_{n,m}, \quad a'_{n,m} = A_n(mh) + B_{n-1}(mh), \quad -N \le n \le N$$

The matrix $a'_{n,m}$ is called the Toeplitz matrix of order 2N+1, where $-N \le m, n \le N$, and the elements of the second matrix are zeros except the elements of the first and last rows . We can evaluate the values of the first row by substituting in $B_{n-1}(mh)$, by $n = -N$, $m = -N + i$, $0 \le i \le 2n$, and the values of the last row by substituting in $A_n(mh)$, by $n = N$, $m = -N + i$.

The **TMM** is said to be convergent of order r in $[-a, a]$, if for N sufficiently large, there exist a constant $D > 0$ independent of N such that

$$\| \phi(x) - \phi_N(x) \| \le D N^{-r} \tag{13}$$

The error term R is determined from the following formula

$$R = \left| \int_{nh}^{nh+h} y^2 k(|x - y|) dy - A_n(x)(nh)^2 - B_n(x)(nh + h)^2 \right| = O(h^3) \tag{14}$$

4. Applications

Example 1:

Consider the integral equation:

$$\varphi(x,t) = f(x,t) + \lambda \int_{-1}^{1} k\,|x - y|\,\varphi(y,t)\,dy + \lambda \int_{0}^{t} \tau^2 \varphi(x,\tau)\,d\tau \tag{15}$$

$$(0 \le t \le T;\ |x| \le 1).$$

Where the exact solution when the kernel takes the logarithmic kernel and Carleman function is $\varphi(x,t) = x^2 + t^2$.

The tables (1-3) contain the numerical results of the example 1 according to different values of time.

1- (T = 0.004, N = 40, K = 3, λ = 0.1627)

X	**Exact**	**Logarithm**	**Err. L.**	**Carleman**	**Err. C.**
-1	1.6000E-05	1.6000E-05	1.396E-11	1.6000E-05	1.7860E-11
-0.8	1.0240E-05	1.0240E-05	1.619E-11	1.0240E-05	1.8000E-11
-0.6	5.7600E-06	5.7600E-06	1.0297E-11	5.7600E-06	1.7484E-11
-0.4	2.5600E-06	2.5600E-06	6.022E-12	2.5600E-06	1.7127E-11
-0.2	6.4000E-07	6.4000E-07	3.9199E-12	6.3998E-07	1.6954E-11
0.2	6.4000E-07	6.4000E-07	3.9199E-12	6.3998E-07	1.6954E-11
0.3	1.4400E-06	1.4400E-06	4.738E-12	1.4400E-06	1.7021E-11
0.7	7.8400E-06	7.8400E-06	1.3216E-11	7.8400E-06	1.7733E-11
0.8	1.0240E-05	1.0240E-05	1.619E-11	1.0240E-05	1.8000E-11
0.9	1.2960E-05	1.2960E-05	1.809E-11	1.2960E-05	1.8180E-11
1	1.6000E-05	1.6000E-05	1.292E-11	1.6000E-05	1.7760E-11

Table (1)

2- (T = 0.2, N = 40, k = 3, λ = 0.16279)

X	**Exact**	**Logarithm**	**Err. L.**	**Carleman**	**Err. C.**
-1	4.0000E-04	4.0001E-04	8.6233E-09	3.9999E-04	1.1259E-08
-0.9	3.2400E-04	3.2401E-04	1.12208E-08	3.2399E-04	1.1444E-08
-0.5	1.0000E-04	1.0000E-04	4.8903E-09	9.9989E-05	1.0824E-08
-0.4	6.4000E-05	6.4004E-05	3.7483E-09	6.3989E-05	1.0719E-08
-0.3	3.6000E-05	3.6003E-05	2.95239E-09	3.5989E-05	1.0647E-08
-0.2	1.6000E-05	1.6002E-05	2.44606E-09	1.5989E-05	1.0600E-08

-0.1	4.0000E-06	4.0022E-06	2.17052E-09	3.9894E-06	1.0575E-08
0.8	2.5600E-04	2.5601E-04	1.00622E-08	2.5599E-04	1.1311E-08
0.9	3.2400E-04	3.2401E-04	1.12241E-08	3.2399E-04	1.1444E-08
1	4.0000E-04	4.0001E-04	7.9737E-09	3.9999E-04	1.1194E-08

Table (2)

3- (T = 0.9, N = 40, k = 3, λ = 0.16279)

X	Exact	Logarithm	Err. L.	Carleman	Err. C.
-1	8.100E-01	8.1565E-01	1.5654E-03	8.1812E-01	6.6882E-03
-0.9	6.561E-01	6.5620E-01	3.1102E-04	6.5 87E-01	6.5951E-04
-0.8	5.184E-01	5.1842E-01	2.9822E-04	5.19 65E-01	6.3307E-04
-0.7	3.969E-01	3.9606E-01	2.5155E-04	2.5659E-01	6.0310E-04
0	0.000E+00	0.0008E-03	9.4788E-04	4.8062E-02	4.8062E-04
0.1	8.1000-02	8.1038E-02	9.6384E-04	-4.0237E-02	4.8337E-04
0.2	3.2400E-02	3.2481E-02	1.0181E-04	3.6755E-02	4.9155E-04
0.3	7.2900E-02	7.2982E-02	1.1282E-04	2.2396E-02	5.0504E-04
0.4	1.2960E-01	1.2679E-01	1.3189E-04	1.243E-02	5.2357E-04
0.8	5.1840E-01	5.1822E-01	2.9824E-04	5.1209E-01	6.3307E-04
0.9	6.5610E-01	6.5622E-01	3.1119E-04	5.9015E-01	6.5952E-04
1	8.1000E-01	8.10056E-01	1.2599E-03	7.4337E-01	6.6628E-03

Table (3)

Example 2:

$$\phi(x,t) = f(x,t) + \lambda \int_{-1}^{1} k\left|x-y\right| \phi(y,t)\,dy + \lambda \int_{0}^{t} \left(t^2 - 2\tau\right)\phi(x,\tau)\,d\tau \qquad (16)$$

$$(0 \le t \le T \;\; ; \; \left|x\right| \le 1).$$

Where the exact solution for the logarithmic kernel and Carleman function is $\varphi(x,t) = x^2 + 5t + 2$. Tables (4-6) contain the numerical results of example 2 for different values of time.

4- (T = .004, N = 40, k = 3, λ = 0.316)

X	Exact	Logarithm	Err. L.	Carleman	Err. C.
-1	1.6000E-05	1.6000E-05	2.7100E-11	1.6000E-05	3.6660E-11
-0.9	1.2960E-05	1.2960E-05	3.5100E-11	1.2960E-05	3.7900E-11
-0.8	1.0240E-05	1.0240E-05	3.1450E-11	1.0240E-05	3.7160E-11
-0.5	4.0000E-06	4.0000E-06	1.5268E-11	4.0000E-06	3.4482E-11
-0.4	2.5600E-06	2.5600E-06	1.1691E-11	2.5600E-06	3.3931E-11
-0.3	1.4400E-06	1.4400E-06	9.1980E-12	1.4400E-06	3.3554E-11
0	0.0000E+00	6.4717E-12	6.4717E-12	-3.3154E-11	3.3154E-11
0.1	1.6000E-07	1.6001E-07	6.7445E-12	1.5997E-07	3.3193E-11
0.5	4.0000E-06	4.0000E-06	1.5268E-11	4.0000E-06	3.4482E-11
0.6	5.7600E-06	5.7600E-06	1.9992E-11	5.7600E-06	3.5231E-11
0.7	7.8400E-06	7.8400E-06	2.5659E-11	7.8400E-06	3.6159E-11
1	1.6000E-05	1.6000E-05	2.5080E-11	1.6000E-05	3.6200E-11

Table (4)

5- (T = 0.02, N = 40, k = 3, λ = 0.316)

X	Exact	Logarithm	Err. L.	Carleman	Err. C.
-1	4.0000E-04	4.0002E-04	1.6739E-08	3.9998E-04	2.3110E-08
-0.9	3.2400E-04	3.2402E-04	2.1781E-08	3.2398E-04	2.3841E-08
-0.6	1.4400E-04	1.4401E-04	1.2424E-08	1.4398E-04	2.2090E-08
-0.4	6.4000E-05	6.4007E-05	7.2760E-09	6.3979E-05	2.1238E-08
-0.3	3.6000E-05	3.6006E-05	5.7310E-09	3.5979E-05	2.0989E-08
-0.2	1.6000E-05	1.6005E-05	4.7482E-09	1.5979E-05	2.0832E-08
-0.1	4.0000E-06	4.0042E-06	4.2133E-09	3.9793E-06	2.0748E-08
0.5	1.0000E-04	1.0001E-04	9.4928E-09	9.9978E-05	2.1601E-08
0.6	1.4400E-04	1.4401E-04	1.2424E-08	1.4398E-04	2.2090E-08
0.7	1.9600E-04	1.9602E-04	1.5941E-08	1.9598E-04	2.2695E-08
1	4.0000E-04	4.0002E-04	1.5478E-08	3.9998E-04	2.2825E-08

Table (5)

6- (T=0.9, N = 40, k = 3, λ = 0.316)

X	Exact	Logarithm	Err. L.	Carleman	Err. C.
-1	8.1000E-01	8.1038E-01	3.0379E-04	8.1066E-01	1.3634E-04
-0.9	6.5610E-01	6.5639E-01	6.0293E-04	5.2008E-01	1.3602E-04
-0.8	5.1840E-01	5.1849E-01	5.7795E-04	5.8873E-01	1.2967E-04
-0.7	3.9690E-01	3.9652E-01	5.9754E-04	5.9745E-01	1.2235E-04
-0.6	2.9160E-01	2.3080E-01	3.9201E-04	3.7634E-01	1.1526E-04
-0.5	2.0250E-01	2.20379E-01	3.1295E-04	3.2042E-02	1.0896E-04
0.2	3.2400E-02	3.2158E-02	1.9758E-04	3.21186E-02	9.6586E-04
0.3	7.2900E-02	7.2910E-02	2.1891E-04	7.2644E-02	9.9544E-04
0.4	1.2960E-01	1.2518E-01	2.5584E-04	1.2591E-02	1.0368E-04
0.8	5.1840E-01	5.18620E-01	5.7798E-04	5.1887E-01	1.2967E-04
0.9	6.5610E-01	6.5643E-01	6.0328E-04	5.2008E-01	1.3602E-04
1	8.1000E-01	8.10428E-01	2.4282E-04	8.1058E-01	1.3542E-04

Table (6)

Conclusion

From the above results we note that:

1. When the function is symmetric with respect to x , the approximate solution by **TMM** also symmetric to sixth decimal.
2. The error function take one form in **TMM** which maximum at the ends when $x=-1$, and $x=1$ and minimum at the middle when $x=0$ special when $n=21$.
3. As the time increases the error increases, while, as N increases the error decreases.
4. The stability of the **TMM** is better to evaluate the approximate solution than the product Nyström method, where the **TMM** and the product Nyström method are considered as the best two numerical methods to solve the singular integral equation numerically, see [9, 10, 14, 15].

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