On Product Groups in which S-semipermutability Is aTransitive Relation

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Abstract. Let G be a finite group. A subgroup H of G is said to be S-semipermutability in G if H permutes with every Sylow p-subgroup of G with $(p, |H|)$. A group G is said to be SBT-group if S-semipermutability is a transitive relation in$G$. In this paper, we investigate the structure of finite groups that are the mutually permutable products of two solvable SBT-subgroups.

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1 Introduction

All groups considered in this paper will be finite. It is known that a group which is the product of two supersolvable groups is not necessarily supersolvable, even if the two factors are normal subgroups of the group. To create intermediate situations it is usual to consider products of groups whose factors satisfies certain relations of permutability. Following Carocca [6], we say that $G = HK$ is the mutually permutable product of H and K if H permutes with every subgroup of K and vice versa. In addition, if every subgroup of H permutes with every subgroup of$K$, we say that the group G is a totally permutable product of H and K. In a seminal paper, Asaad and Shaalan [5] first introduced this property where they study sufficient conditions for totally and mutually permutable products of two supersolvable subgroups to be supersolvable. More precisely, they proved that if G is the mutually permutable product of the supersolvable subgroups H and K and if either the product is totally permutable or H or K is nilpotent, then G is supersolvable. Recently, Alejandre et al. [2] proved that: If $G = HK$ is the mutually permutable product of the supersolvable subgroups H and K such that $Cor_G(H \cap K) = 1$, then G is supersolvable. One of our objectives in this paper is to continue these investigations.

A group G is said to be a T-group if every subnormal subgroup of G is normal in$G$. Such groups were introduced by Gaschütz [8]. A subgroup H of G is said to permutable in G if $HK = KH$ for every subgroup K of G.A group G is said to be a PT-group if every subnormal subgroup of G is permutable in$G$. Solvable PT-groups were studied and classified by Zacher [15]. A subgroup H of G is said to S-permutable in G if it permutes with every Sylow subgroup of G. A group G is said to be a PST-group if every subnormal subgroup of G is S-permutable in G .The structure of solvable PST -groups was determined by Agrawal [1]; see also Asaad and Csörgö. [4]. As a generalizations of permutability and S-permutability, Chen [7] introduced the following concept: A subgroup H of a group G is said to be semipermutable in G if H permutes with every subgroup of G with $(|H|, |K|) = 1$, and S-semipermutability if H permutes with Sylow p-subgroup of G with $(p, |H|) = 1$. It is easy to see that a permutable (respectively, S-permutable) subgroup of G is a semipermutable (respectively, S-semipermutable) subgroup of G. Semipermutability and S-semipermutability, like normality, permutability and S-permutability is not a transitive relation in an arbitrary group. For example, the symmetric group of degree 4,S_4 is a counterexample. A group G is said to be BT-group (respectively, SBT-group) if the semipermutability (respectively, S-semipermutability) property is a transitive, that is, a group G is said to be BT-group (respectively, SBT-group) if H is semipermutable (respectively, S-semipermutable) in K of G and K is semipermutable (respectively, S-semipermutable) in G imply that H is semipermutable (respectively, S-semipermutable) in G: The solvable BT-groups(respectively, solvable SBT-groups) were studied and classified by Wang et al. [13]. The product of two SBT-groups may fail to be a SBT-group. This can be easily seen by examining the symmetric group of degree 4, S_4, which is a product of a SBT-group isomorphic to the symmetric group of degree 3, S_3, and a SBT-group isomorphic to the klien 4-group, V_4, but it is not a

SBT-group. From this example, in this paper, we investigate the following question about factorized groups: Given a factorized group $G = HK$ which is a product of its subgroups $H$and K. What can be said about the structure of G when some information are known about the structures of H and K?

2. Preliminaries

In this section, we give some results that are needed in the sequel:

Lemma 2.1. Let G be a group. The following statements are equivalent:

(1) G is a solvable BT-group;

(2) G is a solvable SBT-group;

(3) every subgroup of G of prime power order is semipermutable in G;

(4) every subgroup of G is semipermutable in G;

(5) every subgroup of G is S-semipermutable in G;

(6) every subgroup of G of prime power order is S-semipermutable in G.

Proof. See [13, Theorem 3.1, p.147].

Lemma 2.2. If G is a solvable SBT-group, then G is super solvable.

Proof. This is an immediate consequence from Lemma 2.1 and [13, Lemma 2.5, p.146].

Lemma 2.3. Let G be a group. Then the following statements are equivalent:

(1) G is a solvable SBT-group;

(2) G is a subgroup closed SBT-group (the group and all its subgroups are SBT-groups);

(3) G is supersolvable SBT-group.

Proof. (1) → (2): This is an immediate consequence from Lemma 2.1 and [13, Corollary 3.2, p.149].

(2) → (3): Suppose that G is a subgroup closed SBT-group. It is enough to show that G is solvable. By induction on the order of G, we can assume that all proper subgroups of G are supersolvable SBT-groups. Cleary, G possesses a normal Sylow p-subgroup P for some p in $\pi(G)$: Hence, by induction on the order of $G, G/P$ and P are supersolvable, whence G is solvable and therefore G is supersolvable by Lemma2.2.

(3) → (1): It is clear.

Lemma 2.4. If G is a finite solvable SBT-group, then G is a PST-group.

Proof. This is an immediate consequence from Lemma 2.1 and [13, Corollary 3.3,p.149].

Lemma 2.5. If G is not a PST-group and all proper subgroups of G are PST-groups (minimal non-PST-group), then $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non normal cyclic Sylow q-subgroup of G for some distinct primes p and q.

Proof. See [4, Corollary 5, p.241].

Lemma 2.6. If H and K are solvable subgroups of a group G with $|G:H| = p$ and $|G:K| = q$, where p and q are distinct primes in $\pi(G)$, then G is solvable.

Proof. See [3,Lemma 10, p.116].

3. Main Results

We need the following result:

Theorem 3.1. Suppose that G is a minimal non-SBT-group. Then $G = PQ$; where P is a normal Sylow p-subgroup of G and Q is a non normal cyclic Sylow q-subgroup of G for some distinct primes p and q;

Proof. Let M be an arbitrary maximal subgroup of G. Since G is a minimal non-SBT-group, it follows that M is a subgroup closed SBT-group and so by Lemma 2.3, M is a solvable SBT-group, whence M is super solvable by Lemma 2.2. By [14, Theorem 2.3, p.11], we have that G is solvable. Hence, all proper subgroups of G are solvable SB-groups and so by Lemma 2.4 that all proper subgroups are PST-groups. If G is a PST-group, then G is a solvable PST-group. By [4, Lemma 1, p.235], G is supersolvable. Hence G possesses a normal Sylow p-subgroup P; where p is the largest prime dividing the order of G: By Schür Zassenhaus. Theorem [9, Theorem 2.1,221], $G = PK$ where K is a p-Hall subgroup of G. By our choice of G, K is a SBT-group. We argue that every subgroup of prime power order of G is S-semipermutable in G. Let L be a subgroup of prime power order of G:If $L \leq P$, then L is a subnormal subgroup of G as $P \lhd G$ and since G is a PST-group, whence $L \lhd G$. Thus L is S-semipermutable in G: If $L \leq K$, then by Lemma 2.1, L is S-semipermutable in K as K is a SBT-group by hypothesis, and since L is permutable with a Sylow p-subgroup of G, we have that L is S-semipermutable in G. Applying Lemma 2.1, we have that G is SBT-group; a contradiction. Thus we may assume that G is not a PST-group. Since all proper subgroups of G are solvable SBT-groups, it follows by Lemma 2.4 that all proper subgroups of G are PST-groups. Then G is a minimal non-PST-group. Hence, by Lemma 2.5 that $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non normal cyclic Sylow q-subgroup of G for some distinct primes p and q. ∎

Now we prove that:

Theorem 3.2. Suppose that a group $G = HK$ the mutually per mutable product of the subgroups H and K such that $(|H|, |K| = 1$. Then G is a solvable SBT-group if and only if H and K are solvable SBT-groups.

Proof. Suppose that H and K are solvable SBT-groups. Then by Lemma 2.2, both H and K are supersolvable and so by [5, Lemma 2.4, p. 318], G is supersolvable. Let M be an arbitrary maximal subgroup of G. Since G is supersolvable, it follows by[14, Theorem 1.7, p.5] that Mhas a prime index in G, say, p in $\pi(G)$.Since $(|H|,|K|=1$, we can assume that p does not divide, say, $|K|$. Let M_1be an p-Hall subgroup of M. By [9, Theorem 4.1, p.231], we have that $K \leq M_1^x$ for some x in G. Since M_1^x has the same properties as M, we can replace M_1^x by M and so we can assume that without loss of generality that $K \leq M$. Since $G = HK$ it follows that $M = K(H \cap M)$. Clearly, $(|K|,|M \cap H|) = 1$. By hypothesis, K is permutable in H and so K is permutable in $H \cap M$, We argue that $H \cap M$ is permutable in K. Let K_1 be a subgroup of K. By hypothesis, K_1H is a subgroup of G.Then $K_1H \cap M = K_1(H \cap M)$ and hence $K_1(H \cap M)$ is a subgroup of G. Thus $H \cap M$ is permutable in K. Thus K and $(H \cap M)$ are mutually permutable subgroups of M. Also, K and $(H \cap M)$ are SBT-solvable subgroups of M by Lemma 2.3. Thus by induction on the order of G, M is a solvable SBT-group and so all proper subgroups of G are solvable SBT-groups.

If G is a SBT-group, we are done. Thus we may assume that G in not a SBT-group. By Theorem 3.1, there exist a normal Sylow p-subgroup P of G and a non normal cyclic Sylow q-subgroup Q of G, where p and q are distinct primes such that $G = PQ$. Let L be a subgroup of G of prime power order. If $L \leq Q$, then L is S-semipermutable in G as P is a normal Sylow p-subgroup of G. If $L \leq P$, then by hypothesis L is permutes with every subgroup of Q, in particular, L permutes with every Sylow q-subgroup of G. Hence, L is S-semipermutable in G. Thus by Lemma 2.1, G is a solvable SBT-group; a contradiction.

Conversely, suppose that G is a solvable SBT-group. Then by Lemma 2.3, H and K are solvable SBT-groups. ■

The hypothesis $(|H|,|K|) = 1$ cannot be omitted from Theorem 3.2 and this easily seen by the following example:

Example 3.3. Consider the group $G = S_3 \times Z_3$, where $S_3 =< x, y | x^3 = y^2 = 1,\ yx = x^2y >$ and $Z_3 =< z | z_3 = 1 >$. It is easily checked that G contains a subgroup of order3 that fails to be S-semipermutable subgroup of G and so $G$is not a SBT-group.

As an immediate consequence of Theorem 3.2, we have that:

Corollary 3.4. Let $G = HK$ be the product of normal subgroups H and K such that $(|H|, |KH) = 1$. Then G is a solvable SBT-group if and only if H and K are solvable SBT-groups.

Proof. Clearly, if H and K are normal subgroups of G, then H and K are mutually permutable and therefore the result follows directly from Theorem 3.2. ■

We can prove the following result:

Theorem 3.5. Let $G = HK$ be the mutually permutable of subgroups H and K. If H is a solvable SBT-subgroup of G and K is a supersolvable subgroup of G, then G is supersolvable.

Proof. Assume that the result is false and let G be a counterexample of minimal order. We claim that $\Phi(G) = 1$. If not, then $\Phi(G) \neq 1$ and so $G/\Phi(G)$ is supersolvable by our choice of G which implies that G is supersolvable, a contradiction. Thus $\Phi(G) = 1$. Clearly, H and K have a Sylow tower property. By[5, Corollary 3.6, p.324], G has a Sylow tower property. Then G has a normal Sylow p-subgroup P and p is the largest prime dividing the order of G. Our choice of G implies that G/P is supersolvable. If $|P| = p$, then G is supersolvable, a contradiction. Thus we may assume that $|P| = p^n, n \geq 2$. since P is normal in G, it follows that $\Phi(P) \leq \Phi(G) = 1$ and so P is elementary abelian. We argue that P is a minimal normal subgroup of G. If not, by Maschke's Theorem [9, Theorem3.2, p.69], $P = V_1 \times V_2$, where V_i is a V-invariant subgroup of $G(i = 1,2)$ and V is a p'-Hall subgroup of G. Since V_i is normal in P, it follows that V_i is normal in $G(i = 1; 2)$. Since $G = G/(V_1 \cap V_2) \subseteq G/V_1 \times G/V_2$, it follows that G is supersolvable, a contradiction. Thus P is a minimal normal subgroup of G and $P = P_1P_2$ where P_1 and P_2 are Sylow p-subgroups of H and K, respectively.

Let M be a proper subgroup of G such that $H \leq M$. Since $G = HK$ and $H \leq M$, it follows that $M = H(K \cap M)$. It is easy to see that the hypotheses of the theorem are inherited over to M .The minimality of G implies that M is supersolvable. Similarly, if $K \leq M$, we have that M is supersolvable.

Now we consider the following cases:

Case1. $1 < P_1 < P$. Let K be a p-Hall subgroup of K. By hypothesis, HK_1 is a proper subgroup of G. Then by the above discussion , HK_1 is super- solvable and so P is normal in HK_1. Also, P is normal in P as P is abelian, and hence it is normal in G. The minimality of P implies that $P_1 = P$, which impossible as $1 < P_1 < P$.

Case 2. $P_1 = P$ and $1 < P_2 < P$. Since H is a SBT-group, it follows by Lemma 2.1 that P_2 is S-semipermutable in H. If $q \neq p$ and Q is any Sylow q-subgroup of H, then P_2Q is a subgroup of H. Also, P_2 is S-permutable in P_2Q and hence P_2 is a subnormal Hall subgroup of P_2Q which implies that $P_2 \lhd P_2Q$ and so $Q \leq N_H(P_2)$. Thus $O^p(H) \leq N_H(P_2)$, where

$O^p(H)$ is the subgroup generated by all p-elements of H. Since $P_2 \lhd P$ and $O^p(H) \leq N_H(P_2)$, we have that $P_2 \lhd H$. Also $P_2 \lhd K$. Thus $P_2 \lhd G$ and this implies that $P_2 = P$ as P is a minimal normal subgroup of G and this is impossible as $1 < P_2 < P$.

Case3. $P_1 = P$ and $P_2 = 1$. Then $P \leq H$ and K is a p-subgroup of G. Since H is a solvable SBT-group by hypothesis, it follows by Lemma 2.2 that H is supersolvable. By [10, Corollary 10.5.2, p.159], H has a normal subgroup N of order p. By hypothesis, NK is a subgroup of G. Clearly, NK is a proper subgroup of G and so NK is supersolvable by the second paragraph of this proof. Hence N is normal in NK. Since $G = HK$, $N \lhd NK$ and $N \lhd H$, we have that $N \lhd G$. By the minimality of P, we have that $|P| = |N| = p$ which is impossible $|P| = p^n$ $n \geq 2$. Similarly, if $P_2 = P$ and $P_1 = 1$, we obtain a contradiction.

Case4. $P_1 = P_2 = P$. Since K is supersolvable, it follows by [10, Corollary 10.5.2, p.159] that K has a normal subgroup L of order p. Clearly, L is subnormal in H as $P_2 = P_1 = P$. Since H is a solvable SBT-group, it follows by Lemma 2.4 that H is a PST-group. Hence, L is S-permutable in H and so $LQ \leq H$ for any Sylow q-subgroup Q of H with $p \neq q$. Clearly, L is a subnormal Hall subgroup of LQ and so $L \lhd LQ$. Thus $LQ \leq N_H(L)$ and hence $Q \leq N_H(L)$. Then $O^p(H) \leq N_H(L)$. Since $L \lhd P_1$, we have that L is normal in H. Thus $L \lhd G$. Once again by the minimality of P, we have that $|P| = |L| = p$ and this is impossible as $|P| = p^n$, $n \geq 2$. This completes the proof of the theorem. ■

If we require that $G = HK$ be the mutually permutable product of the supersolvable subgroups H and K, then G is not necessarily supersolvable.

Example 3.6. Let H be the direct product $< x > \times < y >$, where $|x| = |y| = 5$ .The maps $\alpha: x \to x^2$, $y \to y^{-2}$ and $\beta: x \to y^{-1}$, $y \to x$ are automorphisms of H and generate a subgroup $A \leq Aut(H)$ of order 8(A is isomorphic with the quaterniongroup). Take $G = H[A]$. Then $L =< H, \alpha >$ and $T =< H, \beta >$ are normal subgroups of G and both are supersolvable since L/H and T/H are abelian of exponent $4 = 5 - 1$. Thus, $G = LT$ is the product of normal supersolvable subgroups but is not itself supersolvable, (see[14, p.8]).

As an immediate consequence of Theorem 3.5,we have:

Corollary 3.7. Let $G = HK$.

(1) If H is a normal solvable SBT-subgroup and K is a normal super-solvable subgroup of G, then G is supersolvable.

(2) If H is a normal nilpotent subgroup and K is a normal supersolvable subgroup of G, then G is supersolvable (see [12, p. 129]).

Proof. (1)This is an immediate consequence of Theorem 3.5.

(2) Since H is nilpotent, it follows that H is a solvable SBT-subgroup of G and so by(1) that G is supersolvable.

We prove the following result:

Theorem 3.8. Suppose that H and K are subgroups closed SBT-subgroup of a group G with $|G: H| = p$ and $|G: K| = q$, where p and q are distinct primes. Then G is a subgroup closed SBT-group or $\pi(G) = 2$.

Proof. We prove the theorem by induction on the order of G. By Lemma 2.6, $G$is solvable. Let M be an arbitrary maximal subgroup of G.Therefore by [9, Theorem 1.5, p.219], M has prime power index in G. We argue that M is a subgroup closed SBT-group. If $M$is a conjugate to H nor K, then M is a solvable SBT-group. Thus we may assume that M is neither conjugate to H nor K. Then by [11, Satz 3.9, p.165], $G = MH = MK$. Hence $|G: H| = |M: M \cap H| = p$ and $|G: K| = |M: M \cap K| = q$, where $(M \cap H)$ and $(M \cap K)$ are solvable SBT-groups of M. So by induction on the order of G, M is a subgroup closed SBT-group. Since M is an arbitrary maximal subgroup of G, we have that all proper subgroups of G are SBT-groups. If $G$is a SBT-group, then $G$is a subgroup closed SBT-group. If G is not a SBT-group, then Theorem 3.1 implies that $\pi(G) = 2$. ■

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