$\mathcal{N}$ -Subalgebras of BF–algebras

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Abstract: In this paper, we introduce the notions of $\mathcal{N}$ -sub algebras in BF-algebras and study it in detail.
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1. Introduction

Y. Imai and K. Iseki [5, 6] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [12], J. Neggers and H. S. Kim introduced the notion of B-algebras, which is a generalization of BCK-algebra. Recently, Andrzej Walendziak defined a BF-algebra [14].

The concept of fuzzy sets was first initiated by Zadeh [15]. After the introduction of fuzzy sets by Zadeh [15], there have been a number of generalizations of this fundamental concept. The generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the point $\{1\}$ into the interval [0,1]. In order to provide a mathematical tool to deal with negative information, Jun et al. [10] introduced $\mathcal{N}$ -structures, based on negative-valued functions.

At present, fuzzification ideas have been applied to other algebraic structures such as groups, rings, and so on. The sets of provable formulas in corresponding inference systems from the point of view of uncertain information can be described by fuzzy subsets of those algebraic semantics. The classical two-valued logic has Boolean algebras as an algebraic semantics. Similarly, for important non-classical logics there are algebraic semantics in the form of classes of algebras. Therefore for getting more results we need to study fuzzy algebraic structures in detail.

Jun et al. [10, 11] discussed the notion of $\mathcal{N}$ -structures in BCH/BCK/BCI-algebras and investigated their properties. They introduced the notions of $\mathcal{N}$ -ideals of subtraction algebras.

In the present paper we continue to study BF-algebras and apply the $\mathcal{N}$ -structures to the filter theory in BF-algebras, also investigate the relationship between $\mathcal{N}$ -sub algebras.

Definition 1.1. [14] Let X be a non-empty set with a binary operation $*$ and a constant 0. Then (X,*, 0) is called a BF-algebra if satisfies the following axioms:

(BF1) $x * x = 0$,
(BF2) $x * 0 = x$,
(BF3) $0 * (x * y) = (y * x)$,
for all $x, y \in X$.

Example 1.2 [14] (a) Let $\mathbf{R}$ be the set of real numbers and $A = (\mathbf{R}; *, 0)$ be the algebra with the operation * defined by

$$x * y = \begin{cases} x & \text{if } y = 0, \\ y & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

Then A is a BF-algebra.

(b) Let $A = [0;\infty)$. Define the binary operation * on A as follows: $x * y = |x - y|$, for all $x, y \in A$. Then $(A; *, 0)$ is a BF-algebra.

Proposition 1.3 [14] Let X be a BF-algebra. Then for any x and y in X, the following hold:

(a) $0 * (0 * x) = x$, for all $x \in A$;
(b) if $0 * x = 0 * y$, then x = y for any $x, y \in A$;
(c) if $x * y = 0$, then $y * x = 0$ for any $x, y \in A$.

Definition 1.4 [14] A non-empty subset S of a BF-algebra X is called a sub algebra of X if $x * y \in S$ for any $x, y \in S$.

Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0,1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B.

Denote by $Q(X,[-1,0])$ the collection of functions from a set X to [-1,0]. We say that an element of $Q(X,[-1,0])$ is a negative-valued function from X to [-1,0] (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

2. $\mathcal{N}$-sub algebras on BF-algebras

In what follows, let X denote a BF-algebra and f an $\mathcal{N}$-function on X unless otherwise specified.

Definition 2.1 By a sub algebra of X based on $\mathcal{N}$-function f (briefly, $\mathcal{N}$-sub algebra of X), we mean an $\mathcal{N}$-structure (X,f) in which f satisfies the following assertion:

$$(\forall x, y \in X)(f(x * y) \le \max\{f(x), f(y)\}).$$

Example 2.2 Let $X = \{0,1,2\}$ be a set with the following table.

*	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ is a BF-algebra, but is not a BCH/BCI/BCK-algebra. Define an $\mathcal{N}$-function $f : X \to [-1,0]$ by $f(1) = -0.6$, $f(1) = -0.3$ and $f(2) = -0.1$. Then (X, f) is an $\mathcal{N}$-sub algebra of X. But $\mathcal{N}$-function $g : X \to [-1,0]$ defined by $g(0) = -0.1$, $g(1) = -0.3$ and $g(2) = -0.4$ is not an $\mathcal{N}$-sub algebra because $g(1*1) = g(0) = -0.1 \nleq \max\{g(1), g(1)\} = -0.3.$

Lemma 2.3 Every $\mathcal{N}$-sub algebra (X, f) of X satisfies the following inequalities:

(i) $(\forall x \in X)(f(x) \ge f(0))$.

(ii) $(\forall x \in X)(f(0 * x) \le f(x))$.

Proof. (i) Note that $x * x = 0$ for all $x \in X$. Using (2.1), we have $f(0) = f(x * x) \le \max\{f(x), f(x)\} = f(x)$, for all $x \in X$.

(ii) Let $x \in X$. Then $f(0 * x) \le \max\{f(0), f(x)\}$

$= \max\{f(x * x), f(x)\}$

$\le \max\{\max\{f(x), f(x)\}, f(x)\} = f(x)$.

Proposition 2.4 If an $\mathcal{N}$-sub algebra (X, f) of X satisfies the following inequality:

$(\forall x, y \in X)(f(x * y) \le f(y))$.

Then f is a constant function.

Proof. Let $x \in X$. Using Lemma 2.3, we have $f(x) = f(x * 0) \le f(0)$. It follows that $f(x) = f(0)$, and so f is a constant function.

Proposition 2.5 Let (X, f) be an $\mathcal{N}$-sub algebra of X and $n \in \mathcal{N}$. Then

(i) $f(\prod^{n} x * x) \le f(x)$, for any odd number n,

(ii) $f(\prod^{n} x * x) = f(x)$, for any even number n,

where $\prod^{n} x * x = \overbrace{x * x * \ldots * x}^{n-times}$

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$. for some positive integer k. We prove by induction, definition and above lemma imply that $f(x * x) = f(0) \le f(x)$. Now suppose that $f(\prod^{2k-1} x * x) \le f(x)$. Then by assumption

$f(\prod^{2(k+1)-1} x * x) = f(\prod^{2k+1} x * x)$

$= f(\prod^{2k-1} x * (x * (x * x)))$

$= f(\prod^{2k-1} x * x)$

$\leq f(x)$.

This proves (i). Similarly we can prove (ii).

Theorem 2.6 The family of $\mathcal{N}$-sub algebras of X forms a complete distributive lattice under the ordering of set inclusion $\subset$.

Proof. Let $\{f_i \mid i \in I\}$ be a family of $\mathcal{N}$- sub algebra of X. Since $[-1,0]$ is a completely distributive lattice with respect to the usual ordering in $[-1,0]$, it is sufficient to show that $\cup f_i$ is a $\mathcal{N}$-sub algebra of X. Let $x, y \in X$. Then

$$(\cup f_i)(x*y) = \sup\{f_i(x*y) \mid i \in I\}$$
$$\leq \sup\{\max\{f_i(x), f_i(y)\} \mid i \in I\}$$
$$= \max(\sup\{f_i(x) \mid i \in I\}, \sup\{f_i(y) \mid i \in I\})$$
$$= \max(\cup f_i(x), \cup f_i(y)).$$

Hence $\cup f_i$ is an $\mathcal{N}$-sub algebra of X.

Theorem 2.7 If (X, f) is an $\mathcal{N}$-sub algebra of X, then the set $X_f := \{x \in X \mid f(x) = f(0)\}$ is a sub algebra of X.

Proof. Let $x, y \in X_f$. Then $f(x) = f(0) = f(y)$, and so $f(x*y) \leq \max\{f(x), f(y)\}$
$= \max\{f(0), f(0)\} = f(0)$.
By Lemma 2.3, we get that $f(x*y) = f(0)$ which means that $x*y \in X_f$.

Theorem 2.8 Let M be a (crisp) subset of X. Suppose that (X, f) is an $\mathcal{N}$-sub algebra of X defined by:

$$f(x) = \begin{cases} \alpha & \text{if } x \in M \\ \beta & \text{otherwise} \end{cases}$$

for some $\alpha, \beta \in [-1,0]$ with $\alpha < \beta$. Then (X, f) is an $\mathcal{N}$-sub algebra if and only if M is a sub algebra of X. Moreover, in this case $X_f = M$.

Proof. Let (M, f) be an $\mathcal{N}$-sub algebra. Let $x, y \in X$ be such that $x, y \in M$. Then
$f(x*y) \leq \max\{f(x), f(y)\} = \max\{\alpha, \alpha\} = \alpha$
and so $x*y \in M$.
Conversely, suppose that M is a sub algebra of X and $x, y \in X$.
(i) If $x, y \in M$ then $x*y \in M$, thus $f(x*y) = \alpha = \max\{f(x), f(y)\}$
(ii) If $x \notin M$ or $y \notin M$, then $f(x*y) \leq \beta = \max\{f(x), f(y)\}$ This shows that (M, f) is an $\mathcal{N}$-sub algebra. Moreover, we have
$X_f := \{x \in X \mid f(x) = f(0)\}$
$= \{x \in X \mid f(x) = \alpha\} = M$.

For any $\mathcal{N}$-function f on X and $t \in [-1,0)$, the set $C(f;t) := \{x \in X \mid f(x) \leq t\}$ is called a closed (f,t)-cut of f and the set $O(f;t) := \{x \in X \mid f(x) < t\}$ is called an open (f,t)-cut of f.

It follows easily that for any $\mathcal{N}$-functions f, g on X;
(1) $f \leq g,\ t \in [-1,0] \Rightarrow C(g;t) \subseteq C(f;t)$;
(2) $t_1 \leq t_2,\ t_1, t_2 \in [-1,0] \Rightarrow C(f;t_1) \subseteq C(f;t_2)$;
(3) $f = g \Leftrightarrow C(f;t) = C(g;t)$, for all $t \in [-1,0]$.

Theorem 2.9 The two level sub algebras $C(f, t_1)$, $C(f, t_2)$ (where $t_1 < t_2$) of f are equal if and only if there is no $x \in X$ such that $t_1 < f(x) \leq t_2$.

Proof. Let $C(f, t_1) = C(f, t_2)$ where $t_1 < t_2$ and there exists $x \in X$ such that $t_1 < f(x) \leq t_2$. Then $C(f, t_1)$ is a proper subset of $C(f, t_2)$, which is a contradiction.
Conversely, suppose that there is no $x \in X$ such that $t_1 < f(x) \leq t_2$. If $x \in C(f, t_2)$, then $f(x) \leq t_2$ and so $f(x) \leq t_1$. Therefore

$x \in C(f, t_1)$, thus $C(f, t_2) \subseteq C(f, t_1)$. Hence $C(f, t_1) = C(f, t_2)$.

Theorem 2.10 Let (X, f) be an $\mathcal{N}$-structure of X with the greatest lower bound λ_0. Then the following conditions are equivalent:

(i) (X, f) is an $\mathcal{N}$-sub algebra of X.

(ii) For all $\lambda \in Im(f)$, the non-empty set $C(f, \lambda)$ is a sub algebra of X.

(iii) For all $\lambda \in Im(f) \setminus \lambda_0$, the non-empty set $O(f; \lambda)$ is a sub algebra of X.

(iv) For all $\lambda \in [0,1]$, the non-empty set $O(f; \lambda)$ is a sub algebra of X.

(v) For all $\lambda \in [0,1]$, the non-empty $C(f; \lambda)$ is a sub algebra of X.

Proof. $(i \to iv)$ Let (X, f) be an $\mathcal{N}$-sub algebra of X, $\lambda \in [0,1]$ and $x, y \in O(f; \lambda)$, then we have $f(x * y) \le \max\{f(x), f(y)\} < \max\{\lambda, \lambda\} = \lambda$. Thus $x * y \in O(f; \lambda)$. Hence $O(f; \lambda)$ is a sub algebra of X.

$(iv \to iii)$ It is clear.

$(iii \to ii)$ Let $\lambda \in Im(f)$. Then $C(f; \lambda)$ is a non-empty set. Since $C(f; \lambda) = \bigcap_{\beta > \lambda} O(f; \beta)$, where $\beta \in Im(f) \setminus \lambda_0$. Then by (iii) and Theorem 2.6, $C(f; \lambda)$ is a sub algebra of X.

$(ii \to v)$ Let $\lambda \in [0,1]$ and $C(f; \lambda)$ be non-empty set. Suppose $x, y \in C(f; \lambda)$. Let $\alpha = \max\{f(x), f(y)\}$, it is clear that $\alpha = \max\{f(x), f(y)\} \le \{\lambda, \lambda\} = \lambda$. Thus $x, y \in C(f; \alpha)$ and $\alpha \in Im(f)$, by (ii) $C(f; \alpha)$ is a sub algebra of X, hence $x * y \in C(f; \alpha)$. Then we have $f(x * y) \le \max\{f(x), f(y)\} \le \{\alpha, \alpha\} = \alpha \le \lambda$. Therefore $x * y \in C(f; \lambda)$. Then $C(f; \lambda)$ is a sub algebra of X.

$(v \to i)$ Assume that the non-empty set $C(f; \lambda)$ is a sub algebra of X, for every $\lambda \in [0,1]$. In contrary, let $x_0, y_0 \in X$ be such that $f(x_0 * y_0) > \max\{f(x_0), f(y_0)\}$. Let $f(x_0) = \gamma$, $f(y_0) = \theta$ and $f(x_0 * y_0) = \lambda$. Then $\lambda > \max\{\gamma, \theta\}$. Consider

$$\lambda_1 = \frac{1}{2}(f(x_0 * y_0) + \max\{f(x_0), f(y_0)\})$$

We get that

$$\lambda_1 = \frac{1}{2}(\lambda + \max\{\gamma, \theta\})$$

Therefore $\gamma < \lambda_1 = \frac{1}{2}(\lambda + \max\{\gamma, \theta\} < \lambda$

$$\theta < \lambda_1 = \frac{1}{2}(\lambda + \max\{\gamma, \theta\} < \lambda$$

Hence $\max\{\gamma, \theta\} < \lambda_1 < \lambda = f(x_0 * y_0)$ so that $x_0 * y_0 \notin C(f; \lambda_1)$ which is a contradiction,

Since $f(x_0) = \gamma \le \max\{\gamma, \theta\} < \lambda_1$ $f(y_0) = \theta \le \max\{\gamma, \theta\} < \lambda_1$ imply that $x_0, y_0 \in C(f; \lambda_1)$. Thus $f(x * y) \le \max\{f(x), f(y)\}$, for all $x, y \in X$.

Theorem 2.11 Any sub algebra of X is a level sub algebra of an $\mathcal{N}$-sub algebra of X.

Proof. Let Y be a sub algebra of X, and f be an $\mathcal{N}$-function on X defined by

$$f(x) = \begin{cases} \alpha & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in [-1, 0]$. It is clear that $C(f; \alpha) = Y$.
Let $x, y \in X$. We consider the following cases:
case 1) If $x, y \in Y$, then $x * y \in Y$ therefore $f(x * y) = \alpha = \max\{\alpha, \alpha\} = \max\{f(x), f(y)\}$.

case 2) If $x, y \notin Y$, then $f(x) = 0 = f(y)$ and so

$f(x * y) \leq 0 = \max\{0, 0\} = \max\{f(x), f(y)\}$.

case 3) If $x \in Y$ and $y \notin Y$ (respectively, $x \notin Y$ and $y \in Y$), then $f(x) = \alpha$ and $f(y) = 0$. Thus

$f(x * y) \leq 0 = \max\{\alpha, 0\} = \max\{f(x), f(y)\}$.

Therefore A is an $\mathcal{N}$-sub algebra of X.

3. Conclusion

In the present paper, we have introduced the concept of $\mathcal{N}$-sub algebras of BF-algebras and investigated some of their useful properties. We believe that these results are very useful in developing algebraic structures also these definitions and main results can be similarly extended to some other algebraic systems such as lattices, Lie algebras, semigroups, rings, nearrings and semirings (hemirings). It is our hope that this work would other foundations for further study of the theory of BF-algebras.
In our future study of fuzzy structure of BF-algebras, may be the following topics should be considered:

(1) To establish a $\mathcal{N}$-ideals of BF-algebras;
(2) To consider the structure of quotient BF-algebras by using these $\mathcal{N}$-ideals;
(3) To get more results in $\mathcal{N}$-sub algebras and application.

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