A new modification of the Laplace Adomian decomposition method for system of integral equations

H.O. Bakodah

Department of Mathematics, Sciences Faculty for Girls, King Abdulaziz, University, Jeddah, Saudi Arabia
h.o.bak@hotmail.com; hbakodah@kau.edu.sa

Abstract: In this paper, we present a new modification of the Laplace decomposition method for solving system of integral equations. The technique is described and illustrated with some numerical examples. The results assert that this method is quite accurate.
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1. Introduction

The solution of integral equations have a major role in the fields of science and engineering. A physical event can be modeled by the differential equation, an integral equation or an integro-differential equation or a system of these. There are several numerical method for solving Volterra integral equation systems of the second kind, such as Rung-kutta method[1], expansion method[2], and operational matrices[3]. Farther more, there are several numerical methods for solving Fredholm integral equation systems of the second kind, such as Newton-Taw method[4], homotopy perturbation[5], hat basis[6], and analytic method[7].

Recently a great deal of interest has been focused on the applications of the Adomian decomposition method to solve system of integral equations[8-11]. The Adomian decomposition method provides the solution in a rapid convergent series with computable terms. This method was successfully applied to linear and nonlinear system of integral equations. Different modification of this method to solve nonlinear Fredholm integral equations are given in [12].

The Laplace Adomian decomposition method (LADM) is a combination of ADM and Laplace transform, this method was successfully used for solving differential equations[13,14,15], nonlinear partial differential equations[16], singular integral equation[17,18], and linear and nonlinear partial differential equations[19].

In this work, we will use a new modification of Laplace decomposition method to solve linear and nonlinear system of integral equations, this numerical technique basically illustrates how the Laplace transform may be used to approximate the solutions of the system of integral equations by manipulating the new modified of decomposition method we illustrate this method with the help of several examples.

2. The New modification of Decomposition Method

In this paper we will consider system of nonlinear Volterra equations in the form

$$u(x) = f(x) + \int_a^x k(x-t)\, F(u(t))\, dt \quad (1)$$

Where

$$u(x) = (u_1(x), u_2(x), \ \dots, u_m(x))^T$$

$$k(x,t) = (k_1(x,t), k_2(x,t), \ \dots, k_m(x,t))^T$$

For the linear case, it is assumed that $F(u(t)) = u(t)$.

Consider the i$^{\text{th}}$ equation of (1)

$$u_i(x) = f_i(x) + \int_a^x k_i(x-t)\, F_i(u(t))\, dt \quad (2)$$

The canonical form of the Adomian equations can be written as

$$u_i(x) = f_i(x) + N_i(x) \quad (3)$$

where

$$N_i(x) = N_i(u_1, u_2, ..., u_n) = \int_a^x k_i(x-t)\, F_i(u(t))\, dt \quad (4)$$

To use the Adomian decomposition method, we put $u_i(x) = \sum_{m=0}^{\infty} u_{i,m}(x)$ and $N_i(x) = \sum_{m=0}^{\infty} A_{i,m}$, where $A_{i,m}, m = 0,1,2,...$, are the Adomian polynomials[20]. Hence (3) can be written as

$$\sum_{m=0}^{\infty} u_{i,m}(x) = f_i(x) + \sum_{m=0}^{\infty} A_{i,m}(u_{10}, u_{11}, \dots u_{1m}, \dots u_{n0}, \dots u_{nm}) \quad (5)$$

From (5), we define

$$u_{i,0}(x) = f_i(x)$$

$$u_{i,m+1}(x) = A_{i,m}(u_{10}, \dots u_{1m}, \dots u_{n0}, \dots u_{nm}),\ i = 1,2,\dots n\ ,\ m = 0,1,2,\dots \quad (6)$$

In practice, all terms of the series $u_i(x)=\sum_{m=0}^{\infty}u_{i,m}(x)$ cannot be determined and so we have an approximation of the solution by the following truncated series $\varphi_{i,k}(x)=\sum_{m=0}^{k-1}f_{i,m}(x)$ with

$$\lim_{k\to\infty}\varphi_{i,k}(x)=u_i \tag{7}$$

To determine the Adomian polynomials, we write

$$u_{i\lambda}(x)=\sum_{m=0}^{k-1}u_{i,m}(x)\ \lambda^m \tag{8}$$

and

$$N_{i,\lambda}(u_1,u_2,\ \ldots,u_n)=\sum_{m=0}^{\infty}A_{i,m}\ \lambda^n \tag{9}$$

where λ is a parameter introduced for convenience. From (9), we obtain

$$A_{i,m}(x)=\frac{i}{m!}\left[\frac{d^n}{d\lambda^n}\ N_{i,\lambda}(u_1,u_2,\ldots,u_n)\right]_{\lambda=0} \tag{10}$$

2.1 The Modified Decomposition method

The assumptions made by Adomian[20] were modified by Wazwaz[21], the modified form was established based on the assumption that the function f_i can be divided into two parts, namely f_{0i} and f_{1i}. Under this assumption we set

$$f_i(x)=f_{0i}(x)+f_{1i}(x) \tag{11}$$

Accordingly, a slight variation was proposed only on the components u_{0i} and u_{1i}. The suggestion was that only the part f_{0i} be assigned to the zeroth component u_{0i}, whereas the remaining part f_{1i} be combined with the other terms to define u_{1i}. Consequently the modified recursive relation

$$u_{0i}(x)=f_{0i}(x)$$
$$u_{1i}(x)=f_{1i}(x)-L^{-1}(A_{0i}) \tag{12}$$
$$u_{k+2,i}(x)=-L^{-1}(A_{k+1,i}) \quad , \quad k\geq 0$$

The modification demonstrate a rapid convergence of the series solution if compared with the standard Adomian decomposition method, and it may give the exact solution for nonlinear equations by using two iterations only and without using the so-called Adomian polynomials.

2.2 A New Modification of Adomian decomposition method

The modified decomposition method in subsection **2.1** depends entirely on the proper selection of the functions f_{0i} and f_{1i}. It appears that trials are the only criteria that can be applied so far. In the new modification by Wazwaz [23] we can replace the process of dividing f_i into two components by a series of infinite components. We therefore suggest that f_i be expressed in Taylor series

$$f_i(x)=\sum_{n=0}^{\infty}f_{in}(x) \tag{13}$$

A new recursive relationship expressed in the form

$$u_{0i}(x)=f_{0i}(x)$$
$$u_{i,k+1}(x)=f_{i,k+1}(x)-L^{-1}(A_{ik}) \qquad ,k\geq 0 \tag{14}$$

It is important to note that if f_i consists of one term only, then scheme (14) reduces to relation(6). Moreover if f_i consists of two terms, then relation(14)reduces to the modified relation(12).
If the computation of $N_i(u_1,u_2,\ldots,u_n)$ in equation(10) is very complicated we can consider the Taylor expansion of $N_i(u_1,u_2,\ldots,u_n)$ and consider a few first terms of the expansion and then apply the main idea of the Adomian algorithm.

3 A new Modification of the Laplace Decomposition Method

The method consists of first applying the Laplace transform to both sides of equations in system (1) or (2) gives

$$U(s)=L\{f(x)\}+L\{k(x-t)\}\ L\{F(u(t))\} \tag{15}$$

The Adomian decomposition method defines the solutions $u_i(x)$ by the infinite series

$$U_i(s)=\sum_{n=0}^{\infty}U_{i,n}(s) \qquad i=1,2,3,\ldots m \tag{16}$$

and similarly

$$u_i(x)=\sum_{n=0}^{\infty}u_{i,n}(x) \qquad i=1,2,3,\ldots,m \tag{17}$$

Where the component $U_{i,n}(s)$, $n\geq 0$, $i=1,2,\ldots,m$ will be determined recursively.

The nonlinear terms $F_i(u(x))$ are usually represented by Adomian polynomials.

$$F_i(u(x))=\sum_{n=0}^{\infty}A_{i,n}(x) \quad ,i=1,2,\ldots,m \qquad (18)$$

Substituting (17) and(18) into (15) leads to

$$L\left\{\sum_{n=0}^{\infty}u_{i,n}(x)\right\}=L\{f_i(x)\}+L\{k_i(x-t)\}\ L\left\{\sum_{n=0}^{\infty}A_{i,n}\right\} \quad ,i=1,2,3,\ldots m \qquad (19)$$

the new modification decomposition method introduces the recursive relation

$$U_{i,0}(s)=L\{f_{i,0}(x)\}$$
$$U_{i,k+1}(s)=L\{f_{i,k+1}(x)\}+L\{k_i(x-t)\}\ L\{A_{i,k+1}\},k\geq 0 \quad ,i=1,2,3,\ldots,m \qquad (20)$$

applying the inverse Laplace transform to the first part of (20) gives $u_{i,0}(x)$ that will define $A_{i,0}$. Using $A_{i,0}$ will enable us to evaluate $u_{i,1}(x)$, the determination of $u_{i,0}(x)$ and $u_{i,1}(x)$ leads to the determination of $A_{i,1}$ that will allows us to determine $u_{i,2}(x)$, and so on. Using (17) the series solution follows immediately. In some cases the exact solution in the closed form may also be obtained.

4 Applications

In this section, we use the new Laplace Adomian decomposition method for solving linear and nonlinear system of integral equations.

4.1 The linear system of integral equation

Example(1)

Consider the following system of Volterra integral equation of the second kind with the exact solutions $u_1(x)=x^2+1$ *and* $u_2(x)=1+x-x^3$.

$$u_1(x)=f_1(x)+\int_0^x(x-t)^3\,u_1(t)\,dt+\int_0^x(x-t)^2\,u_2(t)\,dt$$
$$u_2(x)=f_2(x)+\int_0^x(x-t)^4\,u_1(t)\,dt+\int_0^x(x-t)^3\,u_2(t)\,dt \qquad (21)$$

with

$$f_1(x)=1+x^2-\frac{x^3}{3}-\frac{x^4}{3}$$
$$f_2(x)=1+x-x^3-\frac{x^4}{4}-\frac{x^5}{4}-\frac{x^7}{420} \qquad (22)$$

Taking Laplace transform to the system(21) we obtain

$$U_1(s)=L\{f_1(x)\}+L\{(x-t)^3\}U_1(s)+L\{(x-t)^2\}U_2(s)$$
$$U_2(s)=L\{f_2(x)\}+L\{(x-t)^4\}U_1(s)+L\{(x-t)^3\}U_2(s)$$

The modified Adomian decomposition method introduces the recursive relation

$$U_{1,0}(s)=L\{f_{1,0}(x)\}$$
$$U_{2,0}(s)=L\{f_{2,0}(x)\} \qquad (23)$$

and

$$U_{1,k+1}(s)=L\{f_{1,1}(x)\}+L\{(x-t)^3\}U_{1,k}(s)+L\{(x-t)^2\}U_{2,k}(s)$$
$$U_{2,k+1}(s)=L\{f_{2,1}(x)\}+L\{(x-t)^4\}U_{1,k}(s)+L\{(x-t)^3\}U_{2,k}(s)\ ,k\geq 0 \qquad (24)$$

So that, taking the inverse Laplace transform of both sides of relations (23) and (24) gives

$$u_1(x)=\sum_{n=0}^{\infty}u_{1,n}(x)$$
$$u_2(x)=\sum_{n=0}^{\infty}u_{2,n}(x)$$

The efficient of the modified Laplace decomposition method after 10 iterations compare to exact solutions present in Table(1).

The errors $|e_1|=|u_1-u_1^*|$ and $|e_2|=|u_2-u_2^*|$, where u_1, u_2 the exact solutions and u_1^*, u_2^* the approximate solutions by the modified Laplace decomposition method.

Table(1): Numerical result for example(1) with n=10

x	e_1	e_2
0.1	**8.47033E-32**	**4.23516E-33**
0.2	**8.67362E-29**	**8.67362E-30**
0.3	**5.00166E-27**	**7.50248E-28**
0.4	**8.8819E-26**	**1.77637E-26**
0.5	**8.27206E-25**	**2.06798E-25**
0.6	**5.12201E-24**	**1.53655E-24**
0.7	**2.39294E-23**	**8.3748E-24**
0.8	**9.09677E-23**	**3.63834E-23**
0.9	**2.95437E-22**	**1.32925E-22**
1.0	**8.47447E-22**	**4.2362E-22**

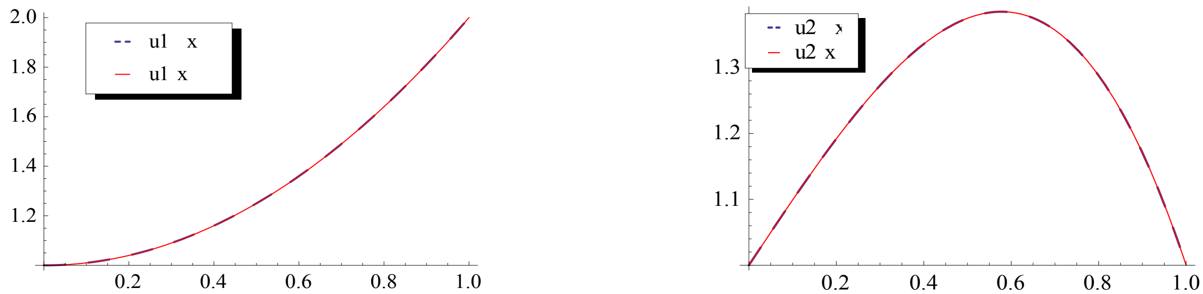

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Fig. 1 Comparison of the exact solution with Adomian decomposition method for Ex.1, (a) for u_1 and (b) for u_2

4.2 The nonlinear system of integral equation Example(2)

Consider the following system of nonlinear Volterra integral equation of the second kind with the exact solutions $u_1(x) = e^{-x}$ *and* $u_2(x) = x$.

$$u_1(x) = f_1(x) + \int_0^x (x-t)^3\, u_1^3(t)\, dt + \int_0^x (x-t)^2\, u_2^2(t)\, dt$$

$$u_2(x) = f_2(x) + \int_0^x (x-t)^4\, u_1^3(t)\, dt + \int_0^x (x-t)^3\, u_2^2(t)\, dt \qquad (25)$$

with

$$f_1(x) = e^{-x} - \tfrac{1}{3}x^3 + \tfrac{1}{3}x^2 - \tfrac{2}{9}x - \tfrac{2}{27}e^{-3x} - \tfrac{1}{30}x^5 + \tfrac{2}{27}$$

$$f_2(x) = x - \tfrac{1}{3}x^4 + \tfrac{4}{9}x^3 - \tfrac{4}{9}x^2 + \tfrac{8}{27}x + \tfrac{8}{81}e^{-3x} - \tfrac{1}{60}x^6 \qquad (26)$$

Taking Laplace transform to the system(26), then to apply the new modification of the Adomian decomposition method, we first set the Taylor expansion for $f_i(x)$, we obtain

$$f_1(x) = 1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 - \tfrac{5}{24}x^4 + \tfrac{13}{120}x^5 - \tfrac{53}{720}x^6 + \tfrac{23}{720}x^7 - \tfrac{97}{8064}x^8 + \tfrac{1457}{362880}x^9 + \ldots$$

$$f_2(x) = x - \tfrac{1}{5}x^5 + \tfrac{1}{12}x^6 - \tfrac{3}{70}x^7 + \tfrac{9}{560}x^8 - \tfrac{3}{560}x^9 + \tfrac{9}{5600}x^{10} + \ldots \qquad (27)$$

Following relation (20) we obtain the numerical results in Table(2) after 10 iterations

The errors $|e_1| = |u_1 - u_1^*|$ and $|e_2| = |u_2 - u_2^*|$, where u_1, u_2 the exact solutions and u_1^*, u_2^* the approximate solutions by the modified Laplace decomposition method.

Table(2): Numerical result for example(2) with n=10

x	e_1	e_2
0.1	3.15203E-15	8.23872E-21
0.2	6.40551E-12	2.82452E-16
0.3	5.40053E-10	1.2703E-13
0.4	1.24457E-8	9.58974E-12
0.5	1.40744E-7	2.6978E-10
0.6	1.01367E-6	4.02932E-9
0.7	5.34607E-6	3.85216E-8
0.8	2.2463E-5	2.62943E-7
0.9	7.95205E-5	1.3701E-6
1.0	2.468722E-4	5.67237E-6

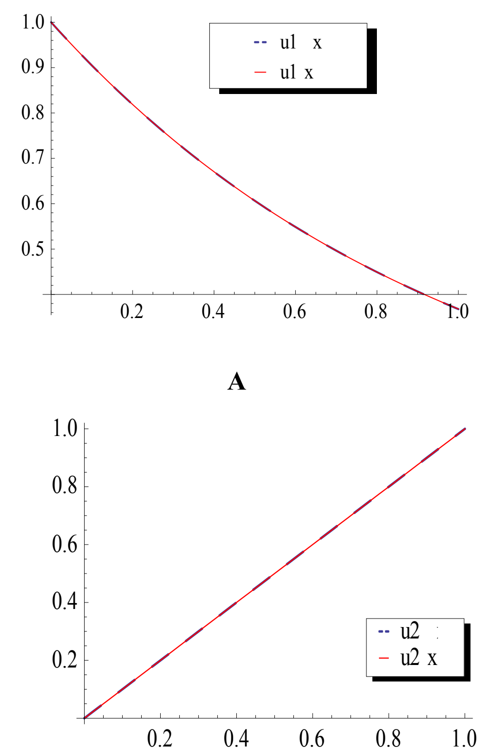

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Fig.2 Comparison of the exact solution with Adomian decomposition method for Ex.2 , (a) for u_1 and (b) for u_2 .

Example (3)

Consider the following system of weakly singular nonlinear Volterra integral equations of the second kind with the exact solutions $u_1(x) = x$ *and* $u_2(x) = \sqrt{x}$.

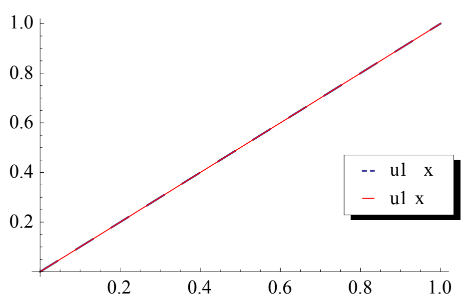
$$u_1(x) = \tfrac{4}{3}x^{\frac{3}{2}} + x + \int_0^x \frac{1}{\sqrt{x-t}}\left(u_2^2(t) - 2u_1(t)\right) dt$$

$$u_2(x) = \tfrac{-16}{15}x^{\frac{5}{2}} + \tfrac{1}{2}\pi\, x + \int_0^x \frac{1}{\sqrt{x-t}}\left(u_1^2(t) - u_2(t)\right) dt$$

The errors $|e_1| = |u_1 - u_1^*|$ and $|e_2| = |u_2 - u_2^*|$ are presented in Table(3), where u_1, u_2 the exact solutions and u_1^*, u_2^* the approximate solutions by the modified Laplace decomposition method with 10 iterations.

Table(3): Numerical result for example(3) with n=10

x	e_1	e_2
0.1	**2.94946E-19**	**1.66326E-17**
0.2	**1.03368E-17**	**3.53027E-17**
0.3	**3.90673E-17**	**6.35044E-17**
0.4	**9.31471E-17**	**1.08449E-16**
0.5	**1.7816E-16**	**1.78105E-16**
0.6	**2.99013E-16**	**2.81258E-16**
0.7	**4.60139E-16**	**4.27486E-16**
0.8	**6.65613E-16**	**6.27121E-16**
0.9	**9.19224E-16**	**8.91203E-16**
1.0	**1.22453E-15**	**1.23144E-15**



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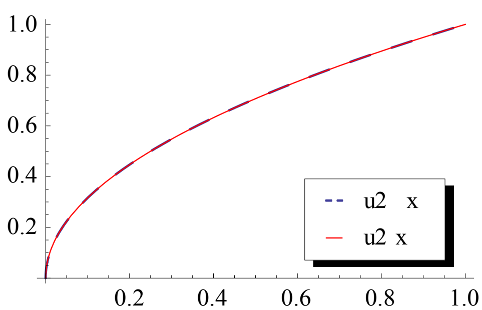

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Fig.3 comparison of the exact solution with Adomian decomposition method for Ex.3 , (a) for u_1 and (b) for u_2 .

5 Conclusion

We have established the approximate solutions for the system of integral equation (linear and nonlinear) by using a new modification of Laplace decomposition method. The new modification combined with Laplace decomposition introduces an efficient algorithm that improves the performance of the standard Adomian decomposition method. As shown in the examples of this paper, the proposed method is a powerful procedure for solving linear and nonlinear system of Volterra integral equations, the numerical results show the high capability of this method compared to other method. The simplicity and also easy to apply in programming are two special features of this method.

Corresponding author
H.O. Bakodah
Department of Mathematics, Sciences Faculty for Girls, King Abdulaziz, University, Jeddah, Saudi Arabia
h.o.bak@hotmail.com hbakodah@kau.edu.sa

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