Deadbeat Response of Nonlinear Systems Described by Discrete-Time *State* Dependent Parameter Using Exact Linearization by Local Coordinate Transformation

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Abstract: This paper considers State Dependent Parameter (SDP), Proportional-Integral-Plus (PIP) control of a wide class of nonlinear systems. Here, the system is modelled using the quasi-linear SDP model structure, in which the parameters are functionally dependent on other variables in the system. This formulation is then be used to design a PIP control law using linear system design strategies, such as pole assignment or suboptimal linear quadratic design. However, since not all feasible SDP model structures can be solved using the basic approach, the present paper develops an *exact linearization by local coordinate transformation* that returns the closed-loop system to a controllable state. Necessary and sufficient conditions are given such that nonlinear SDP systems are feedback equivalent to a controllable linear system. Finally, sufficient simulation examples are illustrated to verify the applicability of the approach. Similar or faster deadbeat response is achieved for nonlinear SDP models having one or more than one input terms respectively. Also, the equivalent linearized model leads to constant state feedback gains that control such nonlinear discrete-time SDP models.
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1. Introduction

Previous papers have introduced the linear Proportional-Integral-Plus (PIP) controller [1-2], in which Non-Minimal State Space (NMSS) models are formulated so that full state variable feedback control can be implemented directly from the measured input and output signals of the controlled process, without resort to the design and implementation of a deterministic state reconstructor or a stochastic Kalman filter. Such PIP control systems have been successfully employed in a wide range of practical applications [3-4].

Typically, however, any inherent nonlinearity in the system has been accounted for in a rather *ad hoc* manner at the design stage, sometimes leading to reduced control performance when applied to particularly difficult, highly nonlinear systems.

One novel research area currently being investigated in order to improve PIP control in such cases, is based on the identification methodology of the State Dependent Parameter (SDP) system. Here, the nonlinear system is modelled using a quasi-linear model structure, in which the parameters are functionally dependent on other variables in the system [5]. In this manner, SDP models can provide a description of a widely applicable class of nonlinear system that even includes chaotic processes and systems that have previously been modelled using a bilinear approach.

The linear-like, ‘affine’ structure of the SDP model means that, at each sampling instant, it can be considered as a ‘frozen’ linear system. This formulation is then used to design an SDP-PIP control law using linear system design strategies such as pole assignment or suboptimal Linear Quadratic (LQ) design, [6-7]. This yields SDP-PIP control systems in which the state feedback gains are themselves state dependent. However, not all feasible SDP model structures are controllable using this basic approach [8-9]. For this reason, the present paper develops an *exact linearization by local coordinate transformation* approach that allows the general discrete-time SDP model form to be controlled. The approach is motivated by conventional *exact linearization via feedback* methods applied to continuous-time systems, [10-13], when these are applied to the special case of discrete-time SDP models.

2. Nonlinear SDP-PIP control

The representation of nonlinear dynamical systems using State Dependent Parameter (SDP) models can be traced to earlier publications such as [14]. However, the practical development of this model is of more recent origin: see [5] and the references therein. In this paper, an SDP model, written in discrete-time incremental form, is considered as:

$$y_k = -a_1(\chi_k)y_{k-1} - \cdots - a_n(\chi_k)y_{k-n} + b_1(\chi_k)u_{k-1} + \cdots + b_m(\chi_k)u_{k-m} \quad (1)$$

where u_k and y_k are the input and output variables respectively. The parameters $a_i(\chi_k)\ \forall 1 \le i \le n$ and $b_j(\chi_k)\ \forall 1 \le j \le m$ are themselves functions of the lagged system variables. In Transfer Function (TF) form, the model (1) becomes,

$$y_k = \frac{b_{1,k+1}z^{-1} + \cdots + b_{m,k+m}z^{-m}}{1 + a_{1,k+1}z^{-1} + \cdots + a_{n,k+n}z^{-n}}u_k = \frac{B_k(z^{-1})}{A_k(z^{-1})}u_k \quad (2)$$

where z^{-1} is the backward shift operator, for which $z^{-i}y(k) = y(k-i)$. Equation (2) alludes to the time variable parameter derivation of the SDP model: see e.g. reference [5] for details. Here, $a_i(\chi_k)\ \forall 1 \le i \le n$ and $b_j(\chi_k)\ \forall 1 \le j \le m$ are functions of the state variables. The backward shift operator notation utilised in (2) suggests that, for example, if $a_{1,k} = a_1(\chi_k) = f(y_{k-1})$ then $a_{1,k+1} = a_1(\chi_{k+1})$ is a function of the un-lagged output y_k.

It is easy to show that model (1) can be represented by the following Non-Minimal State Space (NMSS) form,

$$\mathbf{x}_{k+1} = \mathbf{F}(\chi_{k+1})\mathbf{x}_k + \mathbf{g}(\chi_{k+1})u_k + \mathbf{d}r_{k+1}$$
$$y_k = \mathbf{h}\mathbf{x}_k \quad (3)$$

The $n+m$ dimensional non-minimal state vector $\mathbf{x}_k$ consists of the present and past sampled values of the output and input variables as follows,

$$\mathbf{x}_k = \begin{bmatrix} y_k & y_{k-1} & \cdots & y_{k-(n-1)} & u_{k-1} & \cdots & u_{k-(m-1)} & z_k \end{bmatrix}^T \quad (4)$$

Here, $z_k = z_{k-1} + (r_k - y_k)$ is the integral-of-error state, introduced to ensure inherent type 1 servomechanism performance, where r_k is the reference level. Finally, $\mathbf{F}(\chi_{k+1})$, $\mathbf{g}(\chi_{k+1})$, $\mathbf{d}$ and $\mathbf{h}$ are defined as follows,

$$\mathbf{g}(\chi_{k+1}) = \begin{bmatrix} b_1(\chi_{k+1}) & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & -b_1(\chi_{k+1}) \end{bmatrix}^T$$
$$\mathbf{d} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}^T$$
$$\mathbf{h} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

$$\mathbf{F}(\chi_{k+1}) = \begin{bmatrix} -a_1(\chi_{k+1}) & -a_2(\chi_{k+1}) & \ldots & -a_{n-1}(\chi_{k+1}) & -a_n(\chi_{k+1}) & b_2(\chi_{k+1}) & \ldots & b_{m-1}(\chi_{k+1}) & b_m(\chi_{k+1}) & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ a_1(\chi_{k+1}) & a_2(\chi_{k+1}) & \cdots & a_{n-1}(\chi_{k+1}) & a_n(\chi_{k+1}) & -b_2(\chi_{k+1}) & \ldots & -b_{n-1}(\chi_{k+1}) & -b_m(\chi_{k+1}) & 1 \end{bmatrix} \quad (5)$$

The control law associated with the NMSS model (3) takes the usual State Variable Feedback (SVF) form,

$$u_k = -\mathbf{v}_k\,\mathbf{x}_k \quad (6)$$

where the state variable feedback gain vector is,

$$\mathbf{v}_k = [\,f_{o,k} \quad f_{1,k} \quad \cdots \quad f_{n-1,k} \quad g_{1,k} \quad \cdots \quad g_{m-1,k} \quad -k_{I,k}\,]$$

In more conventional block diagram terms, equation (6) can be implemented as shown in Figure 1, where $M_k(z^{-1})$ and $L_k(z^{-1})$ are defined as follows,

$$L_k(z^{-1}) = f_{o,k} + f_{1,k}z^{-1} + \cdots + f_{n-1,k}z^{-1}$$
$$M_k(z^{-1}) = g_{1,k}z^{-1} + \cdots + g_{m-1,k}z^{-1} \quad (7)$$

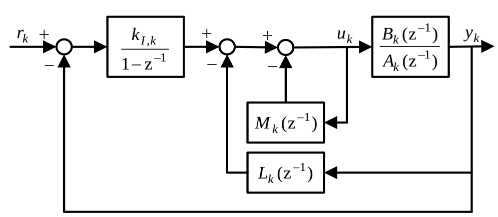


Figure 1. Conventional SDP-PIP control block diagram.

2.1 Nonlinear pole assignment design

In order to develop the pole assignment solution, consider the closed-loop TF, obtained by reducing the block diagram shown in Figure 1 as follows,

$$y_k = \frac{k_{I,k}\,B_k(z^{-1})\ r_k}{\Delta\left[\overline{M}_k(z^{-1})A_k(z^{-1}) + L_k(z^{-1})B_k(z^{-1})\right] + k_{I,k}B_k(z^{-1})} \quad (8)$$

for which $\overline{M}_k(z^{-1}) = 1 + M_k(z^{-1})$, and $\Delta = 1 - z^{-1}$ is the difference operator. Polynomial algebra manipulation of the characteristic equation,

$$\Delta\left[\overline{M}_k(z^{-1})A_k(z^{-1}) + L_k(z^{-1})B_k(z^{-1})\right] + k_{I,k}B_k(z^{-1}) = (1 - p_1 z^{-1})\ldots(1 - p_{n+m}z^{-1}) \quad (9)$$

can be utilized to find the SVF gains at pre-determined pole positions p_i, $\forall\, i = 1,\ldots,n+m$. Nevertheless, it is difficult to obtain closed-form of the gains using direct algebra because the polynomials $A_k(z^{-1})$ and $B_k(z^{-1})$ are most probably function of the un-lagged input u_k as shown in equation (2). However, straightforward iterative procedure can be performed to overcome this difficulty.

Note that, in contrast to linear PIP control [1-2], the nonlinear SDP-PIP case considered here, the gain vector $\mathbf{v}_k$ is itself state dependent. In other words, the control gains are variable and updated *at each sampling instant*.

2.2 Controllability

A prerequisite of global controllability is that the NMSS system (3) is piecewise controllable at each sample *k*. This requirement follows from the fact that if a system is globally controllable, it clearly has to be locally controllable. The NMSS linear controllability conditions are developed by Chotai *et al*. [15]. In the nonlinear case, these are written as follows [9, 16]:

Given a Single-Input Single-Output (SISO) discrete-time system described by the SDP-TF model (2), the NMSS representation, equation (3), as defined by the pair $[\mathbf{F}(\chi_{k+1}), \mathbf{g}(\chi_{k+1})]$, is locally controllable if and only if, the following two conditions are satisfied at each sample:

1. The polynomials $A_k(z^{-1})$ and $B_k(z^{-1})$ are co-prime.
2. $b_{1,k+1}z^{-1} + \cdots + b_{m,k+m}z^{-m} \neq 0$

Nevertheless, it is obvious that, due to the time variation of the parameters in the SDP case, even the conditions 1 and 2 above are not necessarily guaranteed at each sample. In this case, difficulties may arise during the design or implementation of the SDP-PIP algorithm, as illustrated in the example below.

Example (1)

Consider the following nonlinear system [9],

$$y_k = \frac{0.5z^{-1} - (0.4u_k)z^{-2}}{1 - 0.9z^{-1} + 0.08z^{-2}}u_k \quad (10)$$

where y_k is the output variable and u_k is the input variable. It can be seen that at $u_k = 1.25$, the input has no effect on the system output y_k and so the system is uncontrollable due to the invalidation of the second controllability condition. Moreover, at $u_k = 0.125$ and $u_k = 1.0$, the system shows again uncontrollability due to the pole-zero cancellations. Here, the non-minimal state space vector is given by $\mathbf{x}_k = [y_k \;\; y_{k-1} \;\; u_{k-1} \;\; z_k]^T$, while the NMSS model of the system based on equation (3) is defined by,

$$\mathbf{F}(\chi_{k+1}) = \begin{bmatrix} 0.9 & -0.08 & -0.4u_{k-1} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.9 & 0.08 & 0.4u_{k-1} & 1 \end{bmatrix}$$

$$\mathbf{g}(\chi_{k+1}) = [0.5 \;\; 0 \;\; 1 \;\; -0.5]^T$$

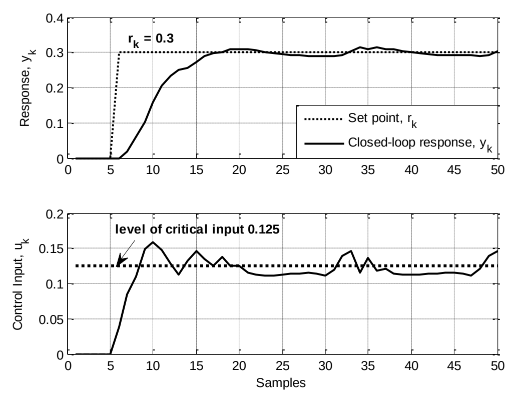
$$\mathbf{d} = [0 \;\; 0 \;\; 0 \;\; 1]^T$$

$$\mathbf{h} = [1 \;\; 0 \;\; 0 \;\; 0]$$

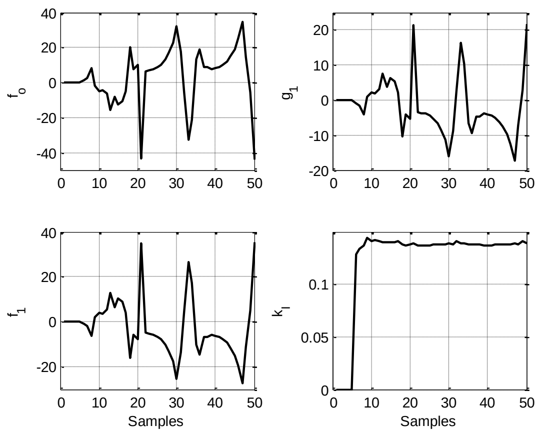
The polynomials in equations (2) and (7) are as follows,

$$\begin{aligned} A_k(z^{-1}) &= 1 + (-0.9)z^{-1} + (0.08)z^{-2} \\ B_k(z^{-1}) &= (0.5)z^{-1} + (-0.4u_k)z^{-2} \\ L_k(z^{-1}) &= f_{o,k} + f_{1,k}z^{-1} \\ M_k(z^{-1}) &= 1 + g_{1,k}z^{-1} \end{aligned} \quad (11)$$

Since the TF model parameter $b_{2,k+2}$ is a function of the present input signal $b_{2,k+2} = -0.4u_k$, it is not a straightforward task to solve the pole assignment problem. However, either algebraic manipulation or an on-line iterative technique can be used to overcome this difficulty. In this manner, Figures 2 and 3 show the closed-loop response for two pole assignment designs.



(a) Upper plot: The set point (dotted line) and the closed-loop response (solid line). Lower plot: The control input (solid line) and the level of critical input 0.125 (dashed line).



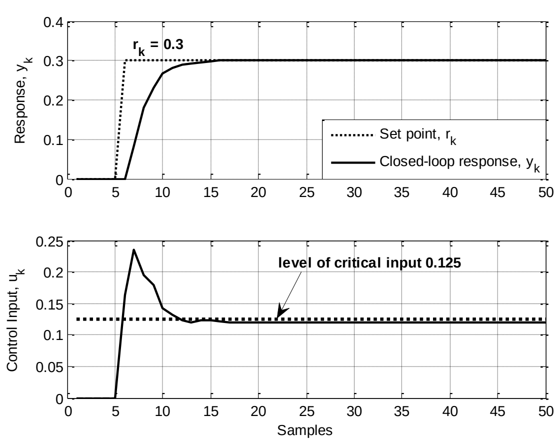
(b) The time varying gains used at poles (0.5).

Figure 2. The closed-loop response of system (10) at set point of 0.3, using poles 0.5 on the complex z-plane.

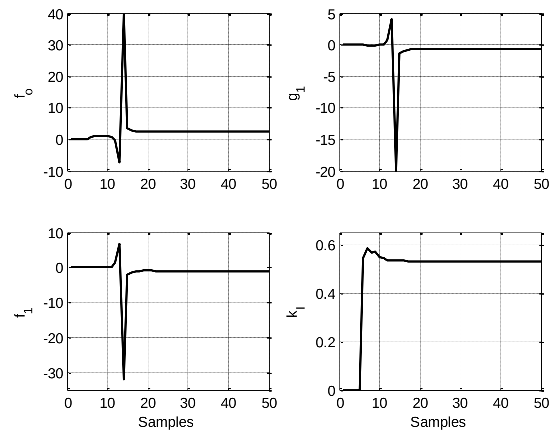
Figure 2 (a) shows an oscillatory closed-loop response for poles placed at 0.5 on the complex z-plane. This happens because the control input approaches one of the critical values, i.e. 0.125, which causes the nonlinear SDP model (10) to start lose its controllability. However, unexpectedly, faster poles placed at 0.3 on the complex z-plane, shows smoother and good tracking response because the control input is now relatively away from this critical value, as illustrated in Figure 3 (a).

Figures 2 (b) and 3 (b) shows the time varying gains used in both cases.

In this example, the minimum absolute value between the control input u_k and its critical level of 0.125 mentioned above is 2.56×10^{-4} in Figure 2 (b). This takes a sufficiently larger value of 1.4×10^{-3} in Figure 3 (b) and hence the closed-loop response avoids the controllability problems observed in the former case. Finally, it is convenient to note that the deadbeat response cannot be achieved at the selected set point, $r_k = 0.3$.



(a) Upper plot: The set point (dotted line) and the closed-loop response (solid line). Lower plot: The control input (solid line)
and the level of critical input 0.125 (dashed line).



(b) The time varying gains used at poles (0.3).

Figure 3. The closed-loop response of the system (8) at set point of 0.3, using poles 0.3 on the complex z-plane.

3. Relative degree of discrete-time systems

The concept of relative degree ρ for continuous-time systems had been proposed in several researches, e.g. [12]. Subsequently, it is suggested for discrete-time SDP models [9, 16], using the analogy between continuous-time and discrete-time systems, which assumes that the notation $\dot{x}$ is equivalent to $\Delta x = x_{k+1} - x_k$.

Relative degree ρ is exactly equal to the number of times one has to differentiate the output y at time t in order to have the value u of the input explicitly appearing [12], i.e. for ubiquitous continuous-time SISO nonlinear system,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \quad (12)$$

It can be said that any SISO nonlinear system has relative degree ρ if,

i. $L_g L_f^k h(x) = 0$ for all x and all $k < \rho - 1$

ii. $L_g L_f^{\rho-1} h(x^{\text{o}}) \neq 0$

Here, $L_f h(x)$ is the derivative of the function *h(x)* along the function $f(x)$, called Lie derivative. The sequence of Lie derivatives is

$$\begin{aligned} L_f^0 h(x) &= h(x) \\ L_f^i h(x) &= \frac{\partial L_f^{i-1} h(x)}{\partial x} f(x) \end{aligned} \quad \forall\, i = 1, 2, \ldots, k \quad (13)$$

Also, the mixed derivatives between $g(x)$ and $f(x)$ are defined as

$$L_g L_f^i h(x) = \frac{\partial L_f^i h(x)}{\partial x} g(x) \qquad \forall\, i = 0, 1, \ldots, k \quad (14)$$

Note that if $L_g L_f^k h(x) = 0$ for all x, then the relative degree cannot be defined, implying that the system output is not affected by the input, i.e. the output depends only on the initial states.

For the purposes of determining the relative degree ρ for discrete-time SDP models, it is essential to express the SDP-NMSS description (3) without including the integral-of-error state. Such a regulator form has the following state vector $\tilde{\mathbf{x}}_k$,

$$\tilde{\mathbf{x}}_k = \begin{bmatrix} y_k & y_{k-1} & .. & y_{k-(n-1)} & u_{k-1} & .. & u_{k-(m-1)} \end{bmatrix}^T \quad (15)$$

The regulator difference form for the description (3) takes the form

$$\begin{aligned} \Delta\tilde{\mathbf{x}}_k &= \left(\tilde{\mathbf{F}}(\chi_{k+1}) - \mathbf{I}\right)\tilde{\mathbf{x}}_k + \tilde{\mathbf{g}}(\chi_{k+1})u_k \\ y_k &= \tilde{\mathbf{h}}\,\tilde{\mathbf{x}}_k \end{aligned} \quad (16)$$

for which $\tilde{\mathbf{g}}(\chi_{k+1})$, and $\tilde{\mathbf{h}}$ are defined as (5) without the last element, and $\tilde{\mathbf{F}}(\chi_{k+1})$ are defined as (5) without the last row, the last column. As shown in (15), the regulator state space vector $\tilde{\mathbf{x}}_k$ is composed of n present and lagged output state variables, and $m-1$ lagged input state variables.

Lemma 1

For discrete-time SDP nonlinear systems,

$$L_f^i h(\tilde{\mathbf{x}}_k) = \Delta^i y_k\,,\ \forall\ i = 0, \ldots, \rho-1 \quad (17)$$

Proof: The proof of this lemma can be obtained from (16) by considering the functions $\mathbf{f}(\tilde{\mathbf{x}}_k) = \left(\tilde{\mathbf{F}}(\chi_{k+1}) - \mathbf{I}\right)\tilde{\mathbf{x}}_k$, $\mathbf{g}(\tilde{\mathbf{x}}_k) = \tilde{\mathbf{g}}(\chi_{k+1})$, and $\mathbf{h}(\tilde{\mathbf{x}}_k) = \tilde{\mathbf{h}}\,\tilde{\mathbf{x}}_k$.

The special nature of difference form of SDP-NMSS description (16) provides that $\mathbf{h}(\tilde{\mathbf{x}}_k) = y_k$, therefore originally, the Lie derivative gives $L_f^0 \mathbf{h}(\tilde{\mathbf{x}}_k) = \Delta^0 y_k = y_k$.

Also, at $\rho > 1$, description (16) ensures the existence of all SDP-TF parameters in function $\mathbf{f}(\tilde{\mathbf{x}}_k)$ which always takes the form

$$\begin{aligned} \mathbf{f}(\tilde{\mathbf{x}}_k) &= \left(\tilde{\mathbf{F}}(\chi_{k+1}) - \mathbf{I}\right)\tilde{\mathbf{x}}_k \\ &= \begin{bmatrix} \Delta y_k & -u_{k-1} & \Delta u_{k-2} & \ldots & \Delta u_{k-(\rho-1)} \end{bmatrix}^T \end{aligned}$$

However, the function $\mathbf{g}(\tilde{\mathbf{x}}_k)$ always takes the form

$$\mathbf{g}(\tilde{\mathbf{x}}_k) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 1 \end{bmatrix}^T$$

Given that $\rho > 1$, the first Lie derivative (13) gives

$$\begin{aligned} L_f \mathbf{h}(\tilde{\mathbf{x}}_k) &= \frac{\partial y_k}{\partial \mathbf{x}_k}\mathbf{f}(\tilde{\mathbf{x}}_k) \\ &= \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}\mathbf{f}(\tilde{\mathbf{x}}_k) \\ &= \Delta y_k \end{aligned}$$

Consecutively, the second Lie derivative (13) gives

$$\begin{aligned} L_f^2 \mathbf{h}(\tilde{\mathbf{x}}_k) &= \frac{\partial \Delta y_k}{\partial \mathbf{x}_k}\mathbf{f}(\tilde{\mathbf{x}}_k) \\ &= \begin{bmatrix} \frac{\partial \Delta y_k}{\partial y_k} & \frac{\partial \Delta y_k}{\partial u_{k-1}} & \frac{\partial \Delta y_k}{\partial u_{k-2}} & \cdots & \frac{\partial \Delta y_k}{\partial u_{k-(\rho-1)}} \end{bmatrix}\mathbf{f}(\tilde{\mathbf{x}}_k) \\ &= \Delta^2 y_k \end{aligned}$$

Eventually, the $\rho - 1$ th Lie derivative gives

$$\begin{aligned} L_f^{\rho-1} \mathbf{h}(\tilde{\mathbf{x}}_k) &= \frac{\partial \Delta^{\rho-1} y_k}{\partial \mathbf{x}_k}\mathbf{f}(\tilde{\mathbf{x}}_k) \\ &= \begin{bmatrix} \frac{\partial \Delta^{\rho-1} y_k}{\partial y_k} & \frac{\partial \Delta^{\rho-1} y_k}{\partial u_{k-1}} & \cdots & \frac{\partial \Delta^{\rho-1} y_k}{\partial u_{k-(\rho-1)}} \end{bmatrix}\mathbf{f}(\tilde{\mathbf{x}}_k) \\ &= \Delta^{\rho-1} y_k \end{aligned}$$

It is now possible to provide the conditions for relative degree ρ for nonlinear discrete-time SDP model (2).

Theorem 1

Nonlinear discrete-time SDP system (2) described in NMSS regulator form (16) has relative degree ρ, if and only if the following conditions are satisfied

$$\begin{aligned} & L_g \Delta^i y_k = 0 \qquad \forall \quad i = 0, \ldots, \rho-2 \\ & L_g \Delta^{\rho-1} y_k \neq 0 \end{aligned} \quad (18)$$

For linear discrete-time TF models, it is straightforward to show that the relative degree ρ is the number of samples required for the input to affect the system following a change in the equilibrium conditions, i.e. the *sampled time delay* δ. Applying the above definition of the time delay to SDP models, it is important to stress that u_k may start to influence the behaviour of the system through one or more of the parameters. For this reason, ρ is not necessarily equal to δ. The following example illustrates this phenomenon.

Example (2)

Consider the following nonlinear discrete-time system, [15]

$$y_k = \alpha\, u_{k-1} y_{k-1}^2 + \beta u_{k-2}^2 \quad (19)$$

One possible formulation for the incremental SDP form (19) is given as

$$y_k = \left(\alpha\, u_{k-1} y_{k-1}\right) y_{k-1} + \left(\beta u_{k-2}\right) u_{k-2} \quad (20)$$

for which its regulator SDP-NMSS description in difference form (16) is given by,

$$\begin{bmatrix} \Delta y_k \\ \Delta u_{k-1} \end{bmatrix} = \begin{bmatrix} \alpha\, u_k y_k - 1 & \beta u_{k-1} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_k \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} y_k \\ u_{k-1} \end{bmatrix}$$

The relative degree ρ of this dynamical system can be obtained through the following differentiation process,

- Given the output y_k ,

$$L_g y_k = \frac{\partial y_k}{\partial \tilde{\mathbf{x}}_k}\mathbf{g}(\tilde{\mathbf{x}}_k)$$
$$= \begin{bmatrix}1 & \alpha\, y_k^2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \alpha\, y_k^2$$

This shows that the relative degree of this system $\rho = 1$ since $L_g \Delta^0 y_k \neq 0$ if and only if $y_k \neq 0$. Here, it is obvious that the construction of the incremental SDP form (20) gives an apparent sample delay $\delta = 2$, i.e. $\rho \neq \delta$. Nevertheless, it is always possible to formulate the difference form of the regulator SDP-NMSS description (16) such that $\rho = \delta$, as illustrated below.

For example (2), another possible formulation for the incremental SDP form (19) is given as

$$y_k = \left(\alpha\, y_{k-1}^2\right)u_{k-1} + \left(\beta\, u_{k-2}\right)u_{k-2} \qquad (21)$$

for which its difference regulator SDP-NMSS form is

$$\begin{bmatrix}\Delta y_k\\ \Delta u_{k-1}\end{bmatrix} = \begin{bmatrix}-1 & \beta u_{k-1}\\ 0 & -1\end{bmatrix}\begin{bmatrix}y_k\\ u_{k-1}\end{bmatrix} + \begin{bmatrix}\alpha y_k^2\\ 1\end{bmatrix}u_k$$

The relative degree ρ of this form can be obtained through the following differentiation process,

- Given the output y_k

$$L_g y_k = \frac{\partial y_k}{\partial \tilde{\mathbf{x}}_k} g(\tilde{\mathbf{x}}_k)$$
$$= \begin{bmatrix}1 & \alpha\, y_k^2\end{bmatrix}\begin{bmatrix}\alpha\, y_k^2\\ 1\end{bmatrix} = 2\alpha\, y_k^2$$

As expected, the relative degree of this description $\rho = 1$ since $L_g \Delta^0 y_k \neq 0$ if and only if $y_k \neq 0$. Therefore, the description (21) gives $\rho = \delta$.

The above example shows that the value of the relative degree ρ represents the number of samples required for the input to affect the response of the dynamic system regardless the construction of the regulator SDP-NMSS form for the dynamic system (19). Nevertheless, the right construction of the incremental SDP form (21) leads certainly to a relative degree ρ equals to the sample delay δ , $\rho = \delta$. This is one important condition for exact linearization.

4. Exact linearization

Exact linearization of nonlinear continuous-time systems has been studied by many authors; e.g. [17-21]. The close analogy between exact linearization by feedback of continuous and discrete time systems has persuaded other authors to apply the technique to discrete systems, e.g. [22-26]. With regard to discrete-time SDP models, an exact linearization has been fully developed and implemented successfully [9, 16], the proposed approach provides the necessary and sufficient conditions for local input-output linearizability. However the deadbeat response using pole placement approach cannot be achieved for systems having $\rho > 2$. For this reason, the present paper develops a local coordinate transformation approach for linearizing discrete-time SDP models; the proposed approach guarantees the deadbeat closed-loop response for the nonlinear systems using time-invariant gains instead. Necessary and sufficient conditions are provided for local input-output linearizability. The approach is applied on sufficient simulation examples. Limitations are also considered with proposed solutions.

The particular nature of the NMSS model (3) prevents the linearization process of the system without changing the construction of the output function. This would violate the requirements of the NMSS model and its associated servomechanism. Therefore, the regulator NMSS form (16) is utilised here for the linearization step.

4.1 Local coordinate transformation

The conditions of relative degree for discrete-time SDP systems defined in Theorem 1, suggest that the functions Δy_k , ..., and $\Delta^{\rho-1} y_k$ have special importance. In fact, they can be used to define a local coordinate transformation around $\tilde{\mathbf{x}}_k^{o}$, for which $\tilde{\mathbf{x}}_k^{o}$ is a point such that $L_g \Delta^{\rho-1} y_k \neq 0$. This is because of the fact that $d(\Delta y_k)$, …, $d(\Delta^{\rho-1} y_k)$ are linearly independent at $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_k^{o}$, for which

$$d(\Delta^i y_k) = \frac{\partial \Delta^i y_k}{\partial x} \qquad \forall\ i = 0, 1, \ldots, \rho-1 \quad (22)$$

Consider the following revised SDP model, in which the relative degree $\rho \geq 1$ is explicitly acknowledged,

$$\begin{aligned} y_k &= -a_1(\chi_k)y_{k-1} - \ldots - a_n(\chi_k)y_{k-n} \\ &\quad + b_\rho(\chi_k)u_{k-\rho} + \ldots + b_m(\chi_k)u_{k-m} \end{aligned} \qquad (23)$$

The SDP-TF form of the model (23) is

$$\begin{aligned} y_k &= \frac{b_\rho(\chi_{k+\rho})\mathrm{z}^{-\rho} + \cdots + b_m(\chi_{k+m})\mathrm{z}^{-m}}{1 + a_1(\chi_{k+1})\mathrm{z}^{-1} + \cdots + a_n(\chi_{k+n})\mathrm{z}^{-n}}u_k \\ &= \frac{B_k(\mathrm{z}^{-1})}{A_k(\mathrm{z}^{-1})}u_k \end{aligned} \qquad (24)$$

Recalling that n is the number of output parameters $a_1(\chi_{k+1})$, ..., $a_n(\chi_{k+n})$, and $p = m - \rho + 1$ is the number of input parameters $b_\rho(\chi_{k+\rho})$, ..., $b_m(\chi_{k+m})$. The SDP-TF form (24) can be described in regulator NMSS difference form (16), for which the regulator state space vector $\tilde{\mathbf{x}}_k$ has length $n+m-1$.

Lemma 2

For nonlinear discrete-time SDP model with n output parameters, and p input parameters, the transformations

$$\Delta Y_{k-(i-1)} = \Delta y_{k-(i-1)} \qquad \forall\ i=1,\dots,n$$
$$\Delta^{j-1}U_{k-(\rho-1)} = \Delta^{j} y_k - \sum_{i=2}^{n}\Delta^{j-1}y_{k-(i-1)} \ \forall\ j=1,\dots,\rho-1 \tag{25}$$

ensure $n+\rho-1$ independent elements of new full set of linearised regulator state space vector $\tilde{\mathbf{X}}_k$, for which

$$\tilde{\mathbf{X}}_k = \begin{bmatrix} Y_k & \dots & Y_{k-(n-1)} & U_{k-1} & \dots & U_{k-(\rho-1)} \end{bmatrix}^T \tag{26}$$

Proof: The first transformation in (25) maps the n output states in the linearised regulator state space vector $\tilde{\mathbf{X}}_k$ as follows

$$Y_{k-(i-1)} = y_{k-(i-1)} \qquad \forall \quad i=1,\dots,n \tag{27}$$

In case of $\rho \geq 2$, the second transformation in (25) maps the rest $\rho-1$ input states in the linearised regulator state space vector $\tilde{\mathbf{X}}_k$. Here, the last or $(\rho-1)^{\text{th}}$ input state, $U_{k-(\rho-1)}$, is obtained by setting $j=1$ as

$$U_{k-(\rho-1)} = \Delta y_k - \sum_{i=2}^{n} y_{k-(i-1)} \tag{28}$$

The preceding $(\rho-2)^{\text{th}}$ linearised input state, $U_{k-(\rho-2)}$, is obtained by setting $j=2$ as

$$\Delta U_{k-(\rho-1)} = \Delta^2 y_k - \sum_{i=2}^{n}\Delta y_{k-(i-1)}$$
$$U_{k-(\rho-2)} - U_{k-(\rho-1)} = \Delta^2 y_k - \sum_{i=2}^{n}\Delta y_{k-(i-1)}$$

therefore

$$U_{k-(\rho-2)} = \Delta^2 y_k - \sum_{i=2}^{n}\Delta y_{k-(i-1)} + U_{k-(\rho-1)} \tag{29}$$

The preceding $(\rho-3)^{\text{th}}$ linearised input state, $U_{k-(\rho-3)}$, is obtained by setting $j=3$, as

$$\Delta^2 U_{k-(\rho-1)} = \Delta^3 y_k - \sum_{i=2}^{n}\Delta^2 y_{k-(i-1)}$$
$$\Delta U_{k-(\rho-2)} - \Delta U_{k-(\rho-1)} = \Delta^3 y_k - \sum_{i=2}^{n}\Delta^2 y_{k-(i-1)}$$
$$U_{k-(\rho-3)} - U_{k-(\rho-2)} - \Delta U_{k-(\rho-1)} = \Delta^3 y_k - \sum_{i=2}^{n}\Delta^2 y_{k-(i-1)}$$

therefore

$$U_{k-(\rho-3)} = \Delta^3 y_k - \sum_{i=2}^{n}\Delta^2 y_{k-(i-1)} + \Delta U_{k-(\rho-1)} + U_{k-(\rho-2)} \tag{30}$$

Consequently, the 1st linearised input state, U_{k-1}, is obtained by setting $j=\rho-1$, as

$$\Delta^{\rho-2}U_{k-(\rho-1)} = \Delta^{\rho-1}y_k - \sum_{i=2}^{n}\Delta^{\rho-2}y_{k-(i-1)}$$
$$\Delta^{\rho-3}U_{k-(\rho-1)} - \Delta^{\rho-3}U_{k-(\rho-1)} = \Delta^{\rho-1}y_k - \sum_{i=2}^{n}\Delta^{\rho-2}y_{k-(i-1)}$$
$$\vdots$$
$$U_{k-1} - \sum_{j=1}^{\rho-2}\Delta^{j-1}U_{k-(j+1)} = \Delta^{\rho-1}y_k - \sum_{i=2}^{n}\Delta^{\rho-2}y_{k-(i-1)}$$

therefore

$$U_{k-1} = \Delta^{\rho-1}y_k - \sum_{i=2}^{n}\Delta^{\rho-1}y_{k-(i-1)} + \sum_{j=1}^{\rho-2}\Delta^{j-1}U_{k-(j+1)} \tag{31}$$

Giving that $\rho \geq 2$, equations (28), (29), (30), and (31) can be used to get the general form for the $m-1$ input states as

$$U_{k-q} = \Delta^{\rho-q}y_k - \sum_{i=2}^{n}\Delta^{\rho-1}y_{k-(i-1)} + \sum_{j=q}^{\rho-2}\Delta^{j-q}U_{k-(j+1)} \tag{32}$$
$$\forall \quad q=1,\dots,\rho-1$$

Equation (27) states that at $i=1$,

$$\Delta Y_k = \Delta y_k \tag{33}$$

Also equation (28) states that at $j=1$,

$$\Delta y_k = U_{k-(\rho-1)} + Y_{k-1} + \dots + Y_{k-(n-1)} \tag{34}$$

Substituting (34) into (33) gives

$$Y_{k+1} - Y_k = U_{k-(\rho-1)} + Y_{k-1} + \dots + Y_{k-(n-1)}$$

Rearranging the above equation and shifting the index backward a unit step leads to an incremental linearised form for the system (24), as follows

$$Y_k = Y_{k-1} + Y_{k-2} + \dots + Y_{k-n} + U_{k-\rho} \tag{35}$$

The corresponding TF of the linearised model (35) using backward shift operator, z^{-1}, is

$$Y_k = \frac{\mathrm{z}^{-\rho}}{1-\mathrm{z}^{-1}-\mathrm{z}^{-2}-\dots-\mathrm{z}^{-n}}U_k \tag{36}$$

Therefore the two transformations in (25) give necessary and sufficient conditions for the linearization process for any nonlinear model (24) with relative degree ρ equals the system time delay ρ, i.e. $\delta=\rho$. The linearised model always has one input parameter. Therefore, the order of the numerator polynomial of TF (36) is $\rho=m$. The resulting linear model (36) has unity output and input parameters.

The linearised unity TF model (36) can be represented using the Non-Minimal State Space (NMSS) form as

$$\begin{aligned}\mathbf{X}_{k+1} &= \mathbf{F}\,\mathbf{X}_k + \mathbf{g}U_k + \mathbf{d}\,r_{k+1} \\ Y_k &= \mathbf{h}\,\mathbf{X}_k\end{aligned} \quad (37)$$

for which $n+\rho$ dimensional non-minimal state vector $\mathbf{X}_k$ takes the form

$$\mathbf{X}_k = [Y_k \;\; Y_{k-1} \;\; \cdots \;\; Y_{k-(n-1)} \;\; U_{k-1} \;\; \cdots \;\; U_{k-(\rho-1)} \;\; Z_k]^T \quad (38)$$

Here, $Z_k = Z_{k-1} + (r_k - Y_k)$ is the integral-of-error state variable, and r_k is the reference level. Note that $Y_k = y_k$, according to equation (27). Finally, for $\rho > 1$, the matrices $\mathbf{F}$, $\mathbf{g}$, $\mathbf{d}$ and $\mathbf{h}$ in NMSS form (37) are defined as follows

$$\boldsymbol{g} = [0 \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1 \;\; 0 \;\; \ldots \;\; 0 \;\; 0]^T$$

$$\boldsymbol{d} = [0 \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1]^T$$

$$\boldsymbol{h} = [1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0]$$

$$\boldsymbol{F} = \begin{bmatrix} 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ -1 & -1 & \cdots & -1 & -1 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix} \quad (39)$$

The linear control law associated with the NMSS model (37) takes the usual state variable feedback form,

$$U_k = -\mathbf{v}\,\mathbf{X}_k \quad (40)$$

where the time-invariant state variable feedback gain vector is,

$$\mathbf{v} = [f_o \;\; f_1 \;\; \cdots \;\; f_{n-1} \;\; g_1 \;\; \cdots \;\; g_{m-1} \;\; -k_I]$$

Theorem 2

The second transformation in (25) at $j=\rho$ leads to the mapping equation between the system input u_k and the linearised system input U_k.

This true since the relative degree ρ is the number of differentiations required to the output y in order to have the value of input u explicitly appearing [12].

Consider the next difference of the second transformation (25), i.e. $j=\rho$, as

$$\Delta^{\rho-1}U_{k-(\rho-1)} = \Delta^{\rho} y_k - \sum_{i=2}^{n} \Delta^{\rho-1} y_{k-(i-1)} \quad (41)$$

The general law for the mapping equation (41) can be obtained by considering the SDP model (24) as

$$y_k = -\sum_{i=1}^{n} a_i(\chi_k) y_{k-i} + \sum_{j=\rho}^{m} b_j(\chi_k) u_{k-j} \quad (42)$$

Shifting the SDP model (42) one step ahead gives

$$\begin{aligned} y_{k+1} = &-a_1(\chi_{k+1}) y_k - \sum_{i=2}^{n} a_i(\chi_{k+1}) y_{k-(i-1)} \\ &+ \sum_{j=\rho}^{m} b_j(\chi_{k+1}) u_{k-(j-1)} \end{aligned} \quad (43)$$

Basically, the first difference for the system output Δy_k is $\Delta y_k = y_{k+1} - y_k$. Also, the second difference is $\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$. Subsequently, the ρ th difference is $\Delta^{\rho} y_k = \Delta^{\rho-1} y_{k+1} - \Delta^{\rho-1} y_k$.

Example (3)

Consider the following nonlinear discrete-time system,

$$\begin{aligned} y_k = &-(\alpha_1 y_{k-1} + \alpha_2) y_{k-1} - (\alpha_3 u_{k-4} + \alpha_4) y_{k-2} \\ &-(\alpha_5 y_{k-3}) y_{k-3} + (\beta_1 u_{k-3} + \beta_2) u_{k-3} \end{aligned} \quad (44)$$

The regulator SDP-NMSS description in difference form for system (44) takes the form of (16) for which the regulator state space vector takes the form

$$\tilde{\mathbf{x}}_k = [y_k \;\; y_{k-1} \;\; y_{k-2} \;\; u_{k-1} \;\; u_{k-2}]^T$$

and the matrices $\tilde{\mathbf{F}}(\chi_{k+1}) - \mathbf{I}$, $\tilde{\mathbf{g}}(\chi_{k+1})$, and $\mathbf{h}$ are,

$$\tilde{\mathbf{g}}(\chi_{k+1}) = [0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]^T$$

$$\mathbf{h} = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$$

$$\tilde{\mathbf{F}}(\chi_{k+1}) - \mathbf{I} = \begin{bmatrix} -a_1(\chi_{k+1}) - 1 & -a_1(\chi_{k+1}) & -a_1(\chi_{k+1}) & 0 & b_3(\chi_{k+1}) \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The Lie derivative of the system output y_k is

$$L_g y_k = \frac{\partial y_k}{\partial \tilde{\mathbf{x}}_k} \tilde{\mathbf{g}}(\chi_{k+1}) = 0$$

The first difference for the system output (44) is

$$\begin{aligned} \Delta y_k &= y_{k+1} - y_k \\ &= -(\alpha_1 y_k + \alpha_2 + 1) y_k - (\alpha_3 u_{k-3} + \alpha_4) y_{k-1} \\ &\quad -(\alpha_5 y_{k-2}) y_{k-2} + (\beta_1 u_{k-2} + \beta_2) u_{k-2} \end{aligned} \quad (45)$$

The Lie derivative of the first difference Δy_k is

$$L_g \Delta y_k = \frac{\partial \Delta y_k}{\partial \tilde{\mathbf{x}}_k} \tilde{\mathbf{g}}(\chi_{k+1}) = 0$$

The second difference for the system output (44) is

$$\begin{aligned}\Delta^2 y_k &= \Delta y_{k+1} - \Delta y_k \\ &= -(2\alpha_1 y_k + \alpha_2 + 1)\Delta y_k - \alpha_1 \Delta y_k^2 - \alpha_3 y_k \Delta u_{k-3} \\ &\quad -(\alpha_3 u_{k-3} + \alpha_4)\Delta y_{k-1} - \alpha_5 \Delta y_{k-2}(y_{k-1} + y_{k-2}) \\ &\quad -(\beta_1 u_{k-2} + \beta_2) u_{k-2} + (\beta_1 u_{k-1} + \beta_2) u_{k-1}\end{aligned} \quad (46)$$

The Lie derivative of the second difference $\Delta^2 y_k$ is

$$\begin{aligned}L_g \Delta^2 y_k &= \frac{\partial \Delta^2 y_k}{\partial \tilde{\mathbf{x}}_k} \tilde{\mathbf{g}}(\chi_{k+1}) \\ &= \frac{\partial \Delta^2 y_k}{\partial u_{k-1}} = \beta_1 u_k + \beta_1 u_{k-1} + \beta_2 \neq 0\end{aligned}$$

Therefore, the construction of the given incremental SDP form (44) gives a relative degree $\rho = 3$, which coincides with the time delay, i.e. $\rho = \delta$. This is because of $L_g \Delta^2 y_k \neq 0$, see equation (18) in Theorem 1.

Equation (25) in Lemma 2, can be used to establish the new coordinate system for the nonlinear system (44) as

$$\begin{aligned}\Delta Y_k &= \Delta y_k \\ \Delta Y_{k-1} &= \Delta y_{k-1} \\ \Delta Y_{k-2} &= \Delta y_{k-2} \\ \Delta U_{k-2} &= \Delta^2 y_k - \Delta y_{k-1} - \Delta y_{k-2} \\ U_{k-2} &= \Delta y_k - y_{k-1} - y_{k-2}\end{aligned} \quad (47)$$

Therefore, the linearised regulator state vector $\tilde{\mathbf{X}}_k$ is

$$\begin{aligned}Y_k &= y_k \\ Y_{k-1} &= y_{k-1} \\ Y_{k-2} &= y_{k-2} \\ U_{k-1} &= \Delta^2 y_k - \Delta y_{k-1} - \Delta y_{k-2} + U_{k-2} \\ U_{k-2} &= \Delta y_k - y_{k-1} - y_{k-2}\end{aligned} \quad (48)$$

The new linearised system can be obtained from the transformation (47) as $\Delta Y_k = y_{k-1} + y_{k-2} + U_{k-2}$ which can be written as

$$Y_k = Y_{k-1} + Y_{k-2} + Y_{k-3} + U_{k-3} \quad (49)$$

Therefore, the nonlinear system (44) can be linearised to take the form of (49) using the transformation (47). The linearised system (49) has a new state space vector $\mathbf{X}_k$, according to equation (38), which is

$$\mathbf{X}_k = [Y_k \quad Y_{k-1} \quad Y_{k-2} \quad U_{k-1} \quad U_{k-2} \quad Z_k]^T \quad (50)$$

It is now possible to represent the linearised system (49) in NMSS form (37). The linear control law associated with the NMSS model takes the usual SVF form,

$$U_k = -\mathbf{v}\mathbf{X}_k \quad (51)$$

where the time-invariant SVF gain vector is,

$$\mathbf{v} = [f_o \quad f_1 \quad f_2 \quad g_1 \quad g_2 \quad -k_I] \quad (52)$$

The control law (51) can be implemented as shown in Figure 4, where

$$\begin{aligned}A(z^{-1}) &= 1 - z^{-1} - z^{-2} - z^{-3} \\ B(z^{-1}) &= z^{-3} \\ L(z^{-1}) &= f_o + f_1 z^{-1} + f_2 z^{-2} \\ \bar{M}(z^{-1}) &= 1 + g_1 z^{-1} + g_2 z^{-2}\end{aligned} \quad (53)$$

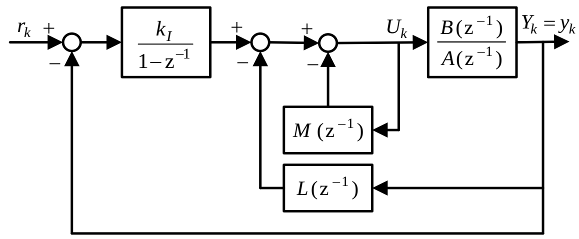


Figure 4. Conventional PIP control block diagram for the linearised system (49).

In order to develop the pole placement solution, consider the closed-loop TF obtained by reducing the block diagram in Figure 4 as follows

$$Y_k = \frac{k_I \, B(z^{-1}) \;\; r_k}{\Delta\left[\bar{M}(z^{-1})\,A(z^{-1}) + L(z^{-1})\,B(z^{-1})\right] + k_I \, B(z^{-1})} \quad (54)$$

Straightforward polynomial algebra manipulation for the characteristic equation,

$$\begin{aligned}&\Delta\left[\bar{M}(z^{-1})\,A(z^{-1}) + L(z^{-1})\,B(z^{-1})\right] + k_I \, B(z^{-1}) \\ &= (1 - p_1 z^{-1}) \ldots (1 - p_{n+m} z^{-1})\end{aligned} \quad (55)$$

can be utilised to find the time-invariant SVF gains defined in (52) at pre-determined pole positions p_i , $\forall \, i = 1, \ldots, 6$.

The mapping between the system input u_k and the linearised system input U_k can be obtained using equation (41) in Theorem 2 as follows

$$\Delta^2 U_{k-2} = \Delta^3 y_k - \Delta^2 y_{k-1} - \Delta^2 y_{k-2} \quad (56)$$

The third difference for the system output (44) can be obtained using equation (46) as

$$\begin{aligned}\Delta^3 y_k &= \Delta^2 y_{k+1} - \Delta^2 y_k \\ &= -(2\alpha_1 y_k + \alpha_2 + 1)\Delta^2 y_k - 2\alpha_1 \Delta y_k (\Delta y_k + \Delta^2 y_k) \\ &\quad - 2\alpha_1 \Delta y_k \Delta^2 y_k - \alpha_1 \Delta^2 y_k^2 - \alpha_3 \Delta y_k \Delta u_{k-2} \\ &\quad - \alpha_3 y_k \Delta^2 u_{k-3} - \alpha_3 \Delta y_k \Delta u_{k-3} - (\alpha_3 u_{k-3} + \alpha_4)\Delta^2 y_{k-1} \\ &\quad - \alpha_5 \Delta y_{k-1}(\Delta y_{k-1} + \Delta y_{k-2}) - \alpha_5 \Delta^2 y_{k-2}(y_{k-1} + y_{k-2}) \\ &\quad - (\beta_1 u_{k-1} + \beta_2) u_{k-1} - (\beta_1 u_{k-1} + \beta_1 u_{k-2} + \beta_2)\Delta u_{k-2} \\ &\quad + (\beta_1 u_k + \beta_2) u_k\end{aligned} \quad (57)$$

Substituting (57) into (56) leads to the quadratic equation,

$$c_1 u_k^2 + c_2 u_k + c_3 = 0 \quad (58)$$

for which

$$c_1 = \beta_1$$
$$c_2 = \beta_2$$
$$\begin{aligned} c_3 = &-\Delta^2 U_{k-2} - \Delta^2 y_{k-1} - \Delta^2 y_{k-2} \\ &-(2\alpha_1 y_k + \alpha_2 + 1)\Delta^2 y_k - 2\alpha_1 \Delta y_k (\Delta y_k + \Delta^2 y_k) \\ &-2\alpha_1 \Delta y_k \Delta^2 y_k - \alpha_1 \Delta^2 y_k^2 - \alpha_3 \Delta y_k \Delta u_{k-2} \\ &-\alpha_3 y_k \Delta^2 u_{k-3} - \alpha_3 \Delta y_k \Delta u_{k-3} - (\alpha_3 u_{k-3} + \alpha_4)\Delta^2 y_{k-1} \\ &-\alpha_5 \Delta y_{k-1}(\Delta y_{k-1} + \Delta y_{k-2}) - \alpha_5 \Delta^2 y_{k-2}(y_{k-1} + y_{k-2}) \\ &-(\beta_1 u_{k-1} + \beta_2)u_{k-1} - (\beta_1 u_{k-1} + \beta_1 u_{k-2} + \beta_2)\Delta u_{k-2} \end{aligned}$$

The solution of (58) maps the linearised input U_k to the system input u_k.

For numerical illustration, assume: $\alpha_1 = 0.1$, $\alpha_2 = 1$, $\alpha_3 = 0.01$, $\alpha_4 = 0.1$, $\alpha_5 = 0.01$, $\beta_1 = 0.15$, and $\beta_2 = 1$. The deadbeat response can be achieved by manipulating the characteristic equation (9) for which

$$A_k(z^{-1}) = 1 + a_{1,k+1} z^{-1} + a_{2,k+2} z^{-2} + a_{3,k+3} z^{-3}$$
$$B_k(z^{-1}) = b_{3,k+3} z^{-3}$$
$$L_k(z^{-1}) = f_{o,k} + f_{1,k} z^{-1} + f_{2,k} z^{-2}$$
$$\overline{M}_k(z^{-1}) = 1 + g_{1,k} z^{-1} + g_{2,k} z^{-2}$$

Similar to Example (1), the parameter $b_{3,k+3} = \beta_1 u_k + \beta_2$ is a function of the present input signal u_k, therefore it is not a straightforward task to solve for the pole placement. However, an iterative technique is developed to solve polynomial (9) at pre-determined poles $p_i = 0$, $\forall\, i = 1,\ldots,6$. This gives six time-variant gains, which form the state variable feedback gain vector,

$$\mathbf{v}_k = [\, f_{o,k} \quad f_{1,k} \quad f_{2,k} \quad g_{2,k} \quad g_{3,k} \quad -k_{I,k} \,] \qquad (59)$$

The closed-loop response of nonlinear system (44) can be achieved by substituting equation (59) into the control law (6). Figure 5 shows an oscillatory deadbeat response due to the existence of nonlinearity in the system. The distortion existed in the deadbeat response arises due to the nonlinearity behaviour of the system. This is one motivation for the linearization process.

However, the six time varying gains used for deadbeat response are depicted in Figure 6.

In the other side, linearization process of the nonlinear system (44) leads to the linear system (49), for which its parameters are: $a_1 = a_2 = a_3 = -1$, and $b_3 = 1$. The deadbeat response can be achieved by manipulating the characteristic equation (55), given the polynomials (53). Direct algebra manipulation for (55) at predetermined poles $p_i = 0$, $\forall\, i = 1,\ldots,6$, gives six linear gains: $f_o = 7$, $f_1 = 6$, $f_2 = 4$, $g_1 = 2$, $g_2 = 4$, and $k_I = 1$. Now, the time-invariant feedback gain vector (52) is used in the linear control law (51) for which the linear state space vector has the form of (50).

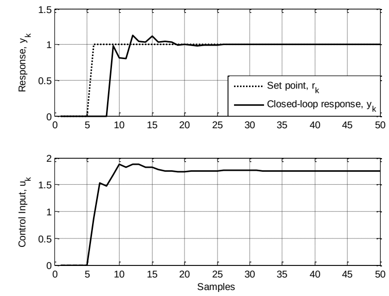


Figure 5. Upper plot: The deadbeat closed-loop response of the nonlinear system (44). Lower plot: The control input.

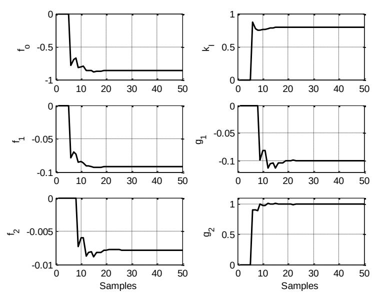


Figure 6. The time varying gains used for the deadbeat response of the nonlinear system (44).

Finally, the linearised control input U_k is mapped to the system control input u_k using the mapping equation (58). The two solutions give a deadbeat response, yet the second solution, which gives positive control input, is selected for comparison with the SDP-PIP control depicted in Figure 5.

The deadbeat closed-loop response of the nonlinear system (44) is depicted in Figure 7. As shown in the figure, the linearization process retains back the standard deadbeat response for the system (44), i.e. no oscillation or overshoot is existed.

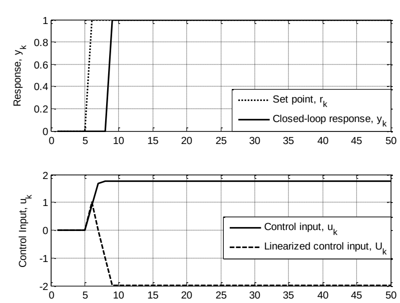


Figure 7. Upper plot: The deadbeat closed-loop response of the nonlinear system (44) based on linearization process. Lower plot: The control input and the linearised control input.

4.2 Exact linearization methodology

Example (3) suggests the following steps for linearization process of any nonlinear SDP system (24):

1. Describe the nonlinear system in the regulator difference form of NMSS description (16). Define the terms $\mathbf{f}(\tilde{\mathbf{x}}_k)$, $\mathbf{g}(\tilde{\mathbf{x}}_k)$, and $\mathbf{h}(\tilde{\mathbf{x}}_k)$, then calculate the relative degree ρ using equation (18) in Theorem 1.
2. Make sure that the relative degree ρ of the nonlinear system coincides with its sample delay δ, $\rho = \delta$. If $\rho \neq \delta$, describe the nonlinear system in another difference form and go to step (1), see Example (2).
3. Define the new linear state space vector $\mathbf{X}_k$ using the transformations (25) in Lemma 2. The elements of the linearised state space vector should take the form of equation (27) for new linearised output states and equation (32) for new linearised input states, i.e.

$$Y_{k-(i-1)} = y_{k-(i-1)} \qquad \forall\, i = 1,\dots,n$$

$$U_{k-q} = \Delta^{\rho-q} y_k - \sum_{i=2}^{n} \Delta^{\rho-1} y_{k-(i-1)} + \sum_{j=q}^{\rho-2} \Delta^{j-q} U_{k-(j+1)}$$

$\forall q = 1, \dots, \rho - 1$

In case of nonlinear systems with unity relative degree, i.e. $\rho = 1$, the second transformation in (25) has no use since no input states are existed in such cases.

4. The new linearised system can be constructed now using transformation (25) at $i = 1$ and $j = 1$,

$$\Delta Y_k = \Delta y_k$$

$$U_{k-(\rho-1)} = \Delta y_k - \sum_{i=2}^{n} y_{k-(i-1)}$$

Therefore, $\Delta Y_k = \sum_{i=2}^{n} Y_{k-(i-1)} + U_{k-(\rho-1)}$. This leads to a linear system with unity parameters at all input and output terms. The incremental form for the linear system is shown in (35), and its corresponding linear TF model is shown in (36).

5. Closed-loop TF or NMSS form is constructed in the usual manner for the linearised system (35). Either pole placement or LQ design is utilised to find the linearised input U_k, where $U_k = -\mathbf{v}\mathbf{X}_k$. It should be noted here that the elements of SVF gain vector $\mathbf{v}$ is constant.
6. Finally, equation (41) in Theorem 2 can then be used to establish the mapping between the linearised input U_k and the actual nonlinear system input u_k.

The procedure for full linearization process is applicable for any SDP nonlinear system. The next example illustrates the above procedure further more.

Example (4)

Consider the system in Example (1) for which its incremental form is

$$y_k = 0.9\,y_{k-1} - 0.08\,y_{k-2} + 0.5u_{k-1} - 0.4u_{k-2}^2 \qquad (60)$$

First, define the nonlinear system (60) in the regulator difference form of NMSS description,

$$\begin{bmatrix} \Delta y_k \\ \Delta y_{k-1} \\ \Delta u_{k-1} \end{bmatrix} = \begin{bmatrix} -0.1 & -0.08 & -0.4u_{k-1} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \\ 1 \end{bmatrix} u_k \qquad (61)$$

Here, the terms $\mathbf{f}(\tilde{\mathbf{x}}_k)$, $\mathbf{g}(\tilde{\mathbf{x}}_k)$, and $\mathbf{h}(\tilde{\mathbf{x}}_k)$ are

$$\mathbf{f}(\tilde{\mathbf{x}}_k) = \begin{bmatrix} -0.1\,y_k - 0.08\,y_{k-1} - 0.4u_{k-1}^2 \\ y_k - y_{k-1} \\ -u_{k-1} \end{bmatrix}$$

$$\mathbf{g}(\tilde{\mathbf{x}}_k) = \begin{bmatrix} 0.5 & 0 & 1 \end{bmatrix}^T$$

$$\mathbf{h}(\tilde{\mathbf{x}}_k) = y_k$$

Second, calculate the relative degree of the nonlinear system (60). The Lie derivative for the output y_k is

$$L_g \Delta^0 y_k = \frac{\partial y_k}{\partial \tilde{\mathbf{x}}_k} g(\tilde{\mathbf{x}}_k) = 0.5 \neq 0$$

This shows that the relative degree of the form (61) is $\rho = 1$, since $L_g \Delta^0 y_k \neq 0$. Also, it is obvious that the time delay δ coincides with the relative degree, i.e. $\rho = \delta = 1$.

Third, define the linear state space vector $\mathbf{X}_k$ using the transformation (25) as

$$\begin{aligned} \Delta Y_k &= \Delta y_k \\ \Delta Y_{k-1} &= \Delta y_{k-1} \end{aligned} \qquad (62)$$

The transformation (62) gives the following elements for the new linearised state space vector,

$$\begin{aligned} Y_k &= y_k \\ Y_{k-1} &= y_{k-1} \end{aligned} \qquad (63)$$

Forth, construct the new linearised system by means of transformation (25) at $i = 1$ and $j = 1$, i.e.

$$\Delta Y_k = \Delta y_k$$

$$U_k = \Delta y_k - y_{k-1}$$

This leads to the following linear system

$$Y_k = Y_{k-1} + Y_{k-2} + U_{k-1} \qquad (64)$$

Fifth, the pole placement solution can be developed by considering the closed-loop TF (54) obtained by reducing the block diagram in Figure 4. Considering the linear system (64), the closed-loop TF is

$$Y_k = \frac{k_I z^{-1}}{1-(2-f_o-k_I)z^{-1}+(f_1-f_o)z^{-2}-(f_1-1)z^{-3}} r_k \quad (65)$$

The deadbeat response can be achieved by direct algebra for the characteristic equation in (65) at predetermined poles $p_i = 0$, $\forall\, i = 1, 2, 3$. This gives three linear gains: $f_o = 1$, $f_1 = 1$, and $k_I = 1$. Now, the time-invariant SVF gain vector (52) is used in the linear control law (51) for which the linear state vector is $\mathbf{X}_k = [Y_k \quad Y_{k-1} \quad z_k]^T$.

Sixth, the mapping between the linearised input U_k and the actual system input u_k is then

$$U_k = \Delta y_k - y_{k-1} \quad (66)$$

Substituting with the value of Δy_k in (66) gives

$$u_k = \frac{U_k + 0.1 y_k + 1.08 y_{k-1} + 0.4 u_{k-1}^2}{0.5} \quad (67)$$

The deadbeat closed-loop response of the nonlinear system (57) is depicted in Figure 8.

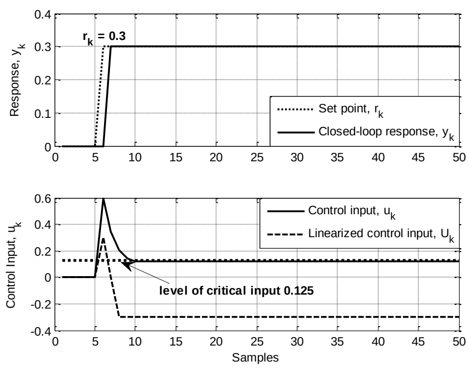


Figure 8. Upper plot: The deadbeat closed-loop response of the nonlinear system (60) based on linearization process. Lower plot: The control input and the linearised control input.

Example (4) shows that the use of exact linearization for SDP nonlinear models sometimes has the advantage of simplifying the model, so reducing the number of gains used in the control law, compared to full SDP-PIP design. In this example, the nonlinear system (60) needs only three linear gains, however in Example (1), the same system needs four time-variant gains.

It is convenient to note that the deadbeat response cannot be reached without linearization. Also the controllability issue shown in Example (1) has been fully avoided using linearization process.

5. Conclusions

This paper develops an 'exact linearization by feedback' approach for the control of a wide range of nonlinear systems. The approach is based on the State Dependent Parameter, Proportional-Integral-Plus (SDP-PIP) control methodology proposed in earlier publications. However, the present paper addresses certain controllability limitations of the basic SDP-PIP algorithm. Necessary and sufficient conditions are given such that the nonlinear SDP systems are feedback equivalent to a controllable linear system.

In particular, by linearizing the SDP model, whilst utilising the SDP-PIP algorithm, any model described by the general SDP structure can now be controlled, and its typical deadbeat response is straightforwardly achievable. Preliminary simulation studies suggest that the new approach not only verifies the standard deadbeat performance of the SDP systems, but also it is generally easy to implement in practice. Moreover, fewer time-invariant input gains are required in case of number of input terms more than unity, i.e. $p \geq 2$. This is because there is always only one linearised input term.

This analysis suggests that, for SDP nonlinear model structures, exact linearization is a very straightforward approach that can be utilised to develop a fixed gain controller. The present examples show that the linearized controller has excellent tracking performance even for deadbeat response compared to the conventional SDP-PIP approach.

In exact linearization by feedback, a modification to the conventional approach is required because of the particular NMSS representation used in SDP-PIP design. Therefore, a regulator form for the NMSS description is introduced.

Furthermore, the term relative degree for discrete-time nonlinear systems has been introduced and fully defined. Also, its importance for the correct description for the NMSS representation is revealed.

Finally, robustness test, input disturbance rejection and output disturbance rejection tests for the linearised system, when applied to real and simulated nonlinear systems, are the subject of current research by the author. Moreover, *on-line* implementation of the linearised controller for practical systems is also being investigated by the author.

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