Explicit series solution of Boussinesq equation by homotopy analysis method

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Abstract: In this paper, the homotopy analysis method (HAM) is applied to solve the Boussinesq equation which is a powerful method for solving the linear and nonlinear partial differential equations. For this purpose, the explicit series solution of the Boussinesq equation is obtained, then the convergence theorem is proved to illustrate the obtained series solution is convergent to the exact solution of the equation. Finally, the proposed method is examined to solve an example.

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1. Introduction

Boussinesq equation which was modeled by Boussinesq is used in costal and ocean engineering and also tidal oscillations [2]. In recent years, some works were done to solve different kinds of Boussinesq equation [1],[3],[4],[6],[9],[10]. In this work, we consider the following Boussinesq equation:

$$u_{tt} + \alpha u_{xx} - u_{xxxx} + \beta(u^2)_{xx} = 0 \qquad (1)$$

with the following initial conditions

$$u(x,0) = f(x), \qquad (2)$$

$$u_t(x,0) = g(x). \qquad (3)$$

Where α,β are constant coefficients and *f, g* are known functions. In this work, we apply the homotopy analysis method (HAM) in order to obtain the solution of the Boussinesq equation. Also, we prove the theorem to illustrate the convergence of this method. At first, in section 2, we remind the main idea of HAM, then in section 3, we apply this method for solving the Boussinesq equation and we prove the convergence theorem. Finally, in section 4, we solve a test example and illustrate the region of convergency by plotting *h*-curves.

2. Preliminaries

In order to describe the HAM [5],[7],[8], we consider the following differential equation:

$$N[u(x,t)] = 0, \qquad (4)$$

where *N* is a nonlinear operator, *x,t* denote the independent variables and *u* is an unknown function. By means of the HAM, we construct the zeroth-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] \quad = qhH(x,t)N[\phi(x,t;q)], \quad (5)$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, *L* is an auxiliary linear operator and *H*(x,*t*) is an auxiliary function. $\phi(x,t;q)$ is an unknown function and $u_0(x,t)$ is an initial guess of *u*(*x, t*). It is obvious that when *q* = 0 and *q* = 1, we have:

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t),$$

respectively. Therefore, as *q* increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the $u_0(x,t)$ to the exact solution $u(x,t)$. By Taylor's theorem, we expand $\phi(x,t;q)$ in a power series of the embedding parameter *q* as follows:

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)\, q^m. \qquad (6)$$

Where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\Big|_{q=0}. \qquad (7)$$

Let the initial guess $u_0(x;t)$, the auxiliary linear operator L, the nonzero auxiliary parameter h and the auxiliary function $H(x;t)$ be properly chosen so that the power series (6) converges at q=1, then, we have:

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t), \qquad (8)$$

which must be one of the solution of the original nonlinear equation. Let the vector

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \dots, u_n(x,t)\}. \qquad (9)$$

By differentiating the zeroth- order deformation (5) m times with respect to the embedding parameter q and then setting q = 0 and finally dividing them by m!, we get the following mth order deformation equation:

$$L[u_m(x,t) - \chi_m u_m(x,t)] = hH(x,t)R_m(\vec{u}_{m-1}) \qquad (10)$$

Where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\Big|_{q=0} \qquad (11)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \qquad (12)$$

It should be emphasized that $u_m(x,t)$ for $m \ge 1$ is governed by the linear equation (10) with linear boundary conditions that come from the original problem.

3. Main Idea

In order to solve the equation (1), we consider the initial approximation $u_0(x,t)$, nonlinear operator

$$N[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2} + \alpha \frac{\partial^2 \phi(x,t;q)}{\partial x^2} - \frac{\partial^4 \phi(x,t;q)}{\partial x^4} + \beta \frac{\partial^2 \phi^2(x,t;q)}{\partial x^2}$$

and the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2},$$

with the property

$$L[c_1(x)t + c_2(x)] = 0,$$

where $c_1(x), c_2(x)$ are the integration constants. Applying (10) under the initial conditions

$$u_m(x,0) = f(x), \qquad \frac{\partial}{\partial t} u_m(x,0) = g(x)$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial t^2} + \alpha \frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{\partial^4 u_{m-1}}{\partial x^4} + \beta \frac{\partial^2 u_{m-1}^2}{\partial x^2}. \qquad (13)$$

The solution of the mth-order deformation equation (10) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\int_0^t \int_0^\tau H(x,\theta)[R_m(\vec{u}_{m-1})]\, d\theta\, d\tau$$

$$+ c_1(x)t + c_2(x), \qquad (14)$$

where the constants $c_1(x), c_2(x)$ are determined by the initial conditions. Then, the series solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \qquad (15)$$

Theorem 3.1. *If the series solution (15) of problem (1) obtained from HAM is convergent then it converges to the exact solution of the problem (1).*

Proof. Let the series $\sum_{m=0}^{+\infty} u_m(x,t)$ be convergent. We assume:

$$u(x,t) = \sum_{m=0}^{+\infty} u_m(x,t),$$

where

$$\lim_{m\to+\infty} u_m(x,t) = 0. \qquad (16)$$

We write

$$\sum_{m=1}^{n}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=$$

$$u_1+(u_2-u_1)+(u_3-u_2)+\cdots+(u_n-u_{n-1})=u_n(x,t),$$

using (16), we have,

$$\sum_{m=1}^{+\infty}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=\lim_{m\to+\infty}u_m(x,t)=0.$$

According to the definition of the operator L, we can write

$$\sum_{m=1}^{+\infty}L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=$$

$$L\sum_{m=1}^{+\infty}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=0.$$

From above expression and equation (10), we obtain

$$\sum_{m=1}^{+\infty}L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hH(x,t)\sum_{m=1}^{+\infty}R_m(\vec{u}_{m-1})=0.$$

Since $h\neq 0$ and $H(x,t)\neq 0$, we have

$$\sum_{m=1}^{+\infty}[R_m(\vec{u}_{m-1})]=0. \qquad (17)$$

From (13), it holds

$$\sum_{m=1}^{+\infty}R_m(\vec{u}_{m-1})=\sum_{m=1}^{+\infty}\frac{\partial^2u_{m-1}}{\partial t^2}+\alpha\sum_{m=1}^{+\infty}\frac{\partial^2u_{m-1}}{\partial x^2}-\sum_{m=1}^{+\infty}\frac{\partial^4u_{m-1}}{\partial x^4}+\sum_{m=1}^{+\infty}\frac{\partial^2u_{m-1}^2}{\partial x^2}=$$

$$\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial t^2}+\alpha\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial x^2}-\sum_{m=0}^{+\infty}\frac{\partial^4u_m}{\partial x^4}+\beta\sum_{m=0}^{+\infty}\frac{\partial^2u_m^2}{\partial x^2}=$$

$$\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial t^2}+\alpha\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial x^2}-\sum_{m=0}^{+\infty}\frac{\partial^4u_m}{\partial x^4}+\beta\frac{\partial^2}{\partial x^2}\sum_{m=0}^{+\infty}u_m^2=$$

$$\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial t^2}+\alpha\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial x^2}-\sum_{m=0}^{+\infty}\frac{\partial^4u_m}{\partial x^4}+\beta\frac{\partial^2}{\partial x^2}[\sum_{m=0}^{+\infty}\sum_{j=0}^{m-1}u_ju_{m-1-j}]=$$

$$\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial t^2}+\alpha\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial x^2}-\sum_{m=0}^{+\infty}\frac{\partial^4u_m}{\partial x^4}+\beta\frac{\partial^2}{\partial x^2}[\sum_{j=0}^{+\infty}\sum_{m=j+1}^{+\infty}u_ju_{m-1-j}]=$$

$$\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial t^2}+\alpha\sum_{m=0}^{+\infty}\frac{\partial^2u_m}{\partial x^2}-\sum_{m=0}^{+\infty}\frac{\partial^4u_m}{\partial x^4}+\beta\frac{\partial^2}{\partial x^2}[\sum_{j=0}^{+\infty}u_j\sum_{i=0}^{+\infty}u_i]=$$

$$\frac{\partial^2u}{\partial t^2}+\alpha\frac{\partial^2u}{\partial x^2}-\frac{\partial^4u}{\partial x^4}+\beta\frac{\partial^2u^2}{\partial x^2}. \qquad (18)$$

From (17) and (18), we have

$$u_{tt}+\alpha u_{xx}-u_{xxxx}+\beta(u^2)_{xx}=0. \quad \square$$

4. Test example

In this section, we solve a Boussinesq equation by applying HAM. The results have been provided by MAPLE.

We consider the following Boussinesq equation

$$u_{tt}-u_{xx}-u_{xxxx}+(u^2)_{xx}=0,$$

with the initial conditions:

$$u(x,0)=\frac{6}{x^2}, \qquad u_t(x,0)=\frac{-12}{x^3}.$$

We choose the initial approximation as

$$u_0(x,t)=\frac{6(x-2t)}{x^3}.$$

Also, we consider

$$H(x,t)\equiv 1.$$

Applying (14), we have:

$$u_1(x,t)=-\frac{6ht^2(3x-4t)}{x^5}+\cdots,$$

$$u_2(x,t)=-\frac{6ht^2(3x^4-4tx^2+3hx^3-4htx^2-5ht^2x+6ht^3)}{x^7}+\cdots,$$

$$u_3(x,t)=\frac{6ht^2(3hx^3+3x^3-4htx^2-4tx^2-7ht^2x+8ht^3)}{x^9}+\cdots,$$

$$\vdots$$

By considering $h=-1$, we have :

$$u_1(x,t) = \frac{18t^2}{x^4} - \frac{24t^3}{x^5} - \frac{504t^4}{x^8},$$

$$u_2(x,t) = \frac{30t^4}{x^6} - \frac{36t^5}{x^7} + \frac{504t^4}{x^8} - \frac{110880t^6}{x^{12}} - \frac{2592t^6}{x^{10}} - \frac{38016t^7}{x^{13}},$$

$$u_3(x,t) = \frac{42t^6}{x^8} - \frac{48t^7}{x^9} + \frac{110880t^6}{x^{12}} + \frac{2592t^6}{x^{10}} + \frac{38016t^7}{x^{13}} - \frac{59875200t^9}{x^{16}} - \frac{7920t^8}{x^{12}} - \frac{965952t^8}{x^{14}} - \frac{30412800t^9}{x^{17}} - \frac{314496t^9}{x^{15}} - \frac{3525120t^{10}}{x^{18}},$$

$\vdots$

Then the series solution expression can be written in the form,

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots.$$

In general, the series solution when $\hbar = -1$ is given by

$$u(x,t) = \frac{6}{x^2} - \frac{12}{x^3}t + \frac{18}{x^4}t^2 - \frac{24}{x^5}t^3 + \frac{30}{x^6}t^4 + \cdots.$$

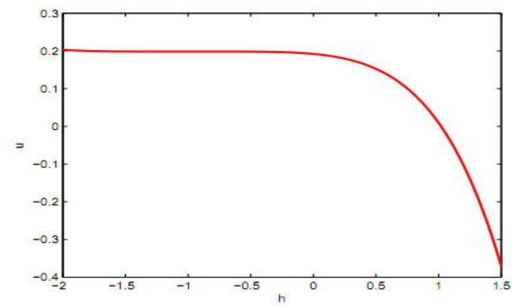
This will, in the limit of infinitely many terms, yields the closed-form solution

$$u(x,t) = \frac{6}{(x+t)^2},$$

which is the exact solution of the equation. In the sequel, we present the *h*-curves to see the convergence region. In figures 1-3, we plot the 3-approximation of $u, u_{xx}, u_{tt}, u_{xxxx}$ and u_{xx}^2 for this example when *x* = 5 and *t* = 0.5 . In the figures, we can see the region of convergence of this method obviously.

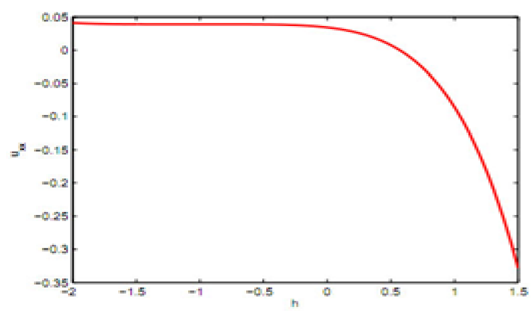
5. Conclusion

In this work, we used the HAM to obtain the explicit series solution of Boussinesq equation, and we proved the convergence theorem to show the method is convergent to the exact solution of this equation. We also plotted the region of convergence via *h*-curves. Therefore, one can apply HAM to solve this equation by an efficient scheme.

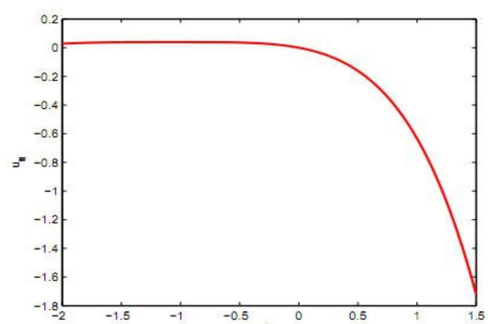


(a)

Figure 1: $\hbar$-curve of u

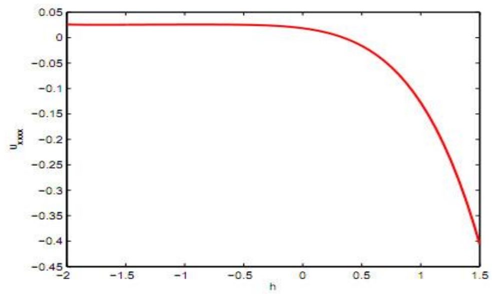


(a) $\hbar$-curve u_{xx}

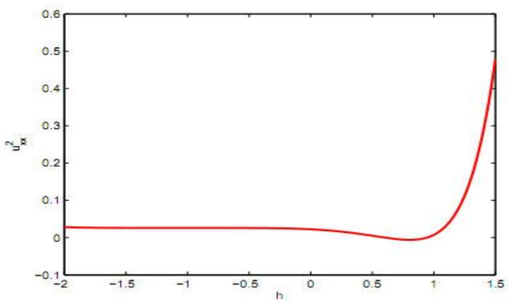


(b) $\hbar$-curve u_{tt}

Figure 2: *h*-curves of u_{xx} and u_{tt}



(a) h-curve u_{xxxx}



(b) h-curve $(u^2)_{xx}$

Figure 3: h-curves of u_{xxxx} and $(u^2)_{xx}$

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