An Enhanced Solution of the Universal Lambert's Problem

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Abstract: In this paper, an iterative method of arbitrary order of convergence $(p \geq 2)$ is developed for solving the universal Lambert's problem using homotopy continuation technique. The method does not need any priori knowledge of the initial guess, a property which avoids the critical situation between divergent to very slow convergent solution, that may exist in the application of other numerical methods depending on initial guess. Computational algorithms and numerical applications will b e applied for some orbits.
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1. Introduction

The basic problems of space dynamics are initial and boundary value problem. In the present paper we shall consider the second one which is known as Lambert's problem.

Lambert's problem was studied in (Escobal 1965, Herrick 1971 and Battin 1964). Also, in 1969 (Lancaster and Blanchard) and (Mansfield 1989) established a unified forms of Lambert's Problem. In 1990, Gooding developed a procedure for the solution. Recently, an algorithm was developed for Batten's method of the universal Lambert's Problem (Alshaary 2008).

The independent variables used in Lambert's Problem satisfied transcendental equation, which is usually solved by iterative methods, which in turn need: (1) initial guess, (2) an iterative scheme. In fact, these two points are not separated from each other, but there is a full agreement that even accurate iterative schemes are extremely sensitive to the initial guess. Moreover, in many cases the initial guess may led to drastic situation between divergent and very slow convergent solutions (Sharaf et al. 2007). In the field of the numerical analysis, very powerful techniques have been devoted(Allgwer and Georg 1990) to solve transcendental equations without any priori knowledge of initial guess these techniques are known as homotopy continuation methods, this technique has been used in initial value problem leading to encouraging results (Alshaary 2003).

In this paper, an iterative method of arbitrary order of convergence $(p \geq 2)$ is developed for solving the universal Lambert's problem using homotopy continuation technique. The method does not need any priori knowledge of the initial guess, a property which avoids the critical situation between divergent to very slow convergent solutions that may exist in the application of other numerical methods depending on initial guess. Computational algorithms and numerical applications will be applied for some orbits.

2. Mathematical Modelling

2.1 Basic equation of Lambert's problem

In Lambert's problem, two positions vectors $r_1 \equiv (x_1, y_1, z_1)$; $r_2 \equiv (x_2, y_2, z_2)$ and the time interval between them Δt which assumed positive are given and its required to find the orbit passing through the two points. Then the quantities of Lambert's problem are; the lengths of the position vectors $|r_1|$, $|r_2|$ and the chord c between the two points, where

$$r_1 = |\boldsymbol{r_1}| = (x_1^2 + y_1^2 + z_1^2)^{\frac{1}{2}},$$

$$r_2 = |\boldsymbol{r_2}| = (x_2^2 + y_2^2 + z_2^2)^{\frac{1}{2}},$$

$$c = |\boldsymbol{r_2} - \boldsymbol{r_1}|$$

$$= \{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}^{\frac{1}{2}}$$

and the related quantities:

$$\lambda_1^2 = r_1 + r_2 + c; \quad \lambda_2^2 = r_1 + r_2 - c.$$

The basic equation of the universal Lambert's theorem could be written as (Vallado and Mcclain 2007).

$$L(z) = C_1 Q_1(z) + C_2 Q_2(z) - \frac{2}{3} C_3 z^{-\frac{3}{2}} - \Delta t = 0, \quad (1)$$

where $z = \frac{1}{a}$ [a is the semi-major axis of the orbit connecting the two points] and Q's are given in terms of the hypergeometric function as:

$$Q_i(z) = F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z\,\lambda_i^2}{4}\right); \; i = 1,2, \quad (2)$$

$$C_1 = \pm\frac{\lambda_1^3}{6K}, \; C_2 = \pm\frac{\lambda_2^3}{6K}, \; K = \sqrt{\mu}, \quad (3)$$

where μ is the gravitational parameter and the upper or lower sign is chosen depending on whether the orbital segment does not (case 1) or does (case 2) respectively include the attracting focus, while

$$C_3 = \begin{cases} 0 & \text{for case1} \\ \frac{-3\pi}{K} & \text{forcase2} \end{cases} \quad (4)$$

2.2 One-point iteration formulae

Let $Y(x) = 0$ such that $Y: R \to R$ smooth map and has a solution $x = \xi$ (say). To construct iterative schemes for solving this equation, will state some basic definition:

1. The error in the k^{th} iterate is defined as
$$\varepsilon_k = \xi - x_k .$$
2. If the sequence $\{x_k\}$ converges to $x = \xi$, then
$$\lim_{k\to\infty} x_k = \xi . \quad (5)$$
3. If there exists a real number $p \geq 1$ such that
$$\lim_{i\to\infty} \frac{|x_{i+1}-\xi|}{|x_i-\xi|^p} = \lim_{i\to\infty} \frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} = L \neq 0. \quad (6)$$
The iterative scheme is of order p at ξ. The constant L is called a asymptotic error constant. For $p = 1$, the convergence is linear; for $p = 2$, the convergence is quadratic; $p = 3, 4, 5$ the convergence is cubic, quadratic and quintic, respectively.
4. We shall consider only stationary one-point iteration formulae which has the form
$$x_{i+1} = R(x_i), \quad i = 0, 1, \ldots \quad (7)$$
5. The order of one point iteration formulae could be determine either from: (a) The Taylor series of the iteration function $R(x_n)$ about ξ (Ralston and Rabinowitz 1978) or from, (b) The Taylor series of the function $Y(x_{k+1})$about x_k (Danby and Burkard 1983).
By the last approach (b), it is easy to form a class of iterative formulae containing members of all integral orders (Sharaf and Sharaf 1998) to solve $Y(x) = 0$ as
$$x_{i+1} = x_i + \sigma_{i,m+2}\,; \quad i = 0,1,2, \ldots \quad m = 0, 1, 2, \ldots, \quad (8)$$
where`
$$\delta_{i,m+2} = \frac{-Y_i}{\sum_{j=1}^{m+1} \left(\delta_{i,m+1}\right)^{j-1} Y_i^{(j)}/j!};$$
$$\delta_{i,1} = 1; \ \forall i \geq 0, \quad (9)$$
$$Y_i^{(j)} = \frac{d^j Y(x)}{dx^j}\Big|_{x=x_i}; \quad Y_i \equiv Y_i^{(0)}. \quad (10)$$

The convergence order is $m + 2$, and given as
$$\epsilon_{i+1} = -\frac{1}{(m+2)!} \frac{Y(\xi)^{(m+2)}}{Y_i^{(1)}(\xi_1)} \epsilon_i^{m+2}, \quad (11)$$
where ξ between x_{i+1} and x_i and ξ_1 between x_{i+1} and ξ.

2.3 Homotopy continuation method for solving $\mathbf{Y(x) = 0}$

Suppose one wishes to obtain a solution of a single non-linear equation in one variable x (say)
$$Y(x) = 0, \quad (12)$$
where $Y: R \to R$ is a mapping which, for our application assumed to be smooth, that is , a map has as many continuous derivatives as requires. Let us consider the situation in which no priori knowledge concerning the zero point of Y is available. Since we assume that such a priori knowledge is not available, then any of the iterative methods will often fail to calculate the zero $\bar{x}$, because poor starting value is likely to be chosen. As a possible remedy, one defines a homotopy or deformation $H: R \times R \to R$ such that
$$H(x, 1) = Q(x); \ H(x, 0) = Y(x),$$
where $Q: R \to R$ is a (trivial) smooth map having known zero point and H is also smooth. Typically, one may choose a convex
$$H(x, \lambda) = \lambda\, Q(x) + (1 - \lambda)\, Y(x), \quad (13)$$
and attempt to trace an implicity defined curve $\Phi(z) \in H^{-1}(0)$ from a starting point $(x_1, 1)$ to a solution point $(\bar{x}, 0)$. If this succeeds, then a zero point $\bar{x}$ of Y is obtained. The curve $\Phi(z) \in H^{-1}(0)$ can be traced numerically if it is parameterized with respect to the parameter λ, then the classical embedding methods can be applied (Algower and George 1990).

3. Computational Developments

3.1 Recurrence formulae for the nth. derivative of $\mathbf{L(z)}$

For the iterative formulae of Equation (1), we establish for the nth derivative of $L(z)$ the following recurrence formula:
$$2zL^{(n+1)}(z) = 3C_1 G_1^{(n)}(z) + 3C_2 G_2^{(n)}(z) - (2n + 3)L^{(n)}, \quad (14)$$
where the G's functions satisfy the recurrence formulae:
$$G_i^{(n)}(z) = 0.125(2n-1)\lambda_i^2 \left(G_i^{(0)}(z)\right)^2 G_i^{(n-1)}(z)\,;$$
$$i = 1,2; \ m = 1,2, \ldots, k, \ G_i^{(0)}(z) = \left(1 - \frac{z\lambda_i^2}{4}\right)^{\frac{-1}{2}}. \quad (15)$$

3.2 Computational Algorithms

Two algorithms are established in this section:

- The first one is for tracing the curve $\Phi(z) \in H^{-1}(0)$ from $\lambda = 1$ to $\lambda = 0$.
- The second one is for solving the basic equation of universal Lambert problem (1).

3.2.1 Computational Algorithm 1

- **Purpose:** To solve $Y(x) = 0$by embedding method.
- **Input:** (1) The function $Q(x)$ with defined root x_1 such that $H(x_1, 1) = 0$
(2) Positive integer m.
- **Output:** Solution x of $Y(x) = 0$.
- **Computational Sequence:**
  1. Set $x = x_1$, $\lambda = (m-1)/m$, $\Delta\lambda = 1/m$
  2. For $i = 1$ to m do
  Begin

Solve $H(y, \lambda) = \lambda\, Q(y) + (1 - \lambda)\, Y(y) = 0,$

iteratively for using x as starting value

$x = y; \quad \lambda = \lambda - \Delta\lambda.$

End.

3.2.2 Computational Algorithm 2

- **Purpose:** To solve the basic equation of Lambert's theorem by iterative schemes of quadratic up to l th convergence orders without priori knowledge of the initial guess using homotopy continuation method with $Q(z) = z + 1$.
- **Input:** $r_1, r_2, c, \Delta t, l, m$ (positive integer), Tol (specified tolerance), μ.
- **Output:** The solution z of the basic equation of the universal Lambert's theorem.
- **Computational steps:**

1. Set $z = -1; \ \Delta\lambda = \frac{1}{m}; \ \lambda = 1 - \Delta\lambda;$

$\lambda_1 = \sqrt{r_1 + r_2 + c}; \ \lambda_2 = \sqrt{r_1 + r_2 - c}$

$C_1 = \pm\frac{\lambda_1^3}{6K}; \ C_2 = \mp\frac{\lambda_2^3}{6K}; \ K = \sqrt{\mu};$

$$C_3 = \begin{cases} 0 & for\ case 1 \\ \frac{-3\pi}{K} & for\ case 2 \end{cases}$$

2. For $i := 1$ to m

Begin {i}

$q = 1 - \lambda; \ A_1 = F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z}{4} * \lambda_1^2\right);$

$A_2 = F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{z}{4} * \lambda_2^2\right)$

$u = z^{-\frac{5}{2}}; \ FV = C_1 * A_1 + C_2 * A_2 - \frac{2}{3} * z * C_3 * u - \Delta t$

$L = \lambda * (z + 1) + q * FV;$

$S_1 = \left(1 - \frac{z}{4} * \lambda_1^2\right)^{-\frac{1}{2}}; \ S_2 = \left(1 - \frac{z}{4} * \lambda_2^2\right)^{-\frac{1}{2}}$

$Q_1 = \frac{3}{2z}(S_1 - A_1); \ Q_2 = \frac{3}{2z}(S_2 - A_2);$

$F[1] = C_1 Q_1 + C_2 Q_2 + C_3 Q_3.$

$Fa[1] = \lambda + q\, F[1]; \quad \Delta z = -\frac{L}{Fa[1]};$

If $l = 2$ go to step 5

$G_1 = 0.125\, \lambda_1^2\, S_1^3; \ G_2 = 0.125\, \lambda_2^2\, S_2^3$

$F[2] = \frac{3}{2z}[3C_1G_1 + 3C_2G_2 - 5\, F[1]];$

$Fa[2] = q\, F[2]$

$H = Fa[1] + \Delta z\, \frac{Fa[2]}{2}; \ \Delta z = -\frac{L}{H}$

If $l = 3$ go to step 3

For $k = 1,4$

Do: $H = Fa[1]; \ B = 1; \ n = k - 1$

$G_1 = 0.125\, (2n - 3)\lambda_1^2\, S_1^2 G_1;$

$G_2 = 0.125\, (2n - 3)\lambda_2^2\, S_2^2 G_2.$

$$F[n] = \frac{1}{2z}[3C_1G_1 + 3C_2G_2 - (2n + 1)F[n - 1]];$$

$$Fa[n] = q\, F[n]$$

For $j: 1$ to $n - 1$ Do: $B = \Delta z \frac{B}{j+1}; \ H = H + B\, Fa[j + 1]$

end {j} ; $\Delta z = -\frac{L}{H};$ end {k}

3. $z = z + \Delta z; \quad \lambda = \lambda - \Delta\lambda$

If $|\Delta z| \le$ Tol go to step 4;

go to step 2.

4. End

Finally, the accuracy of the computations could be checked by the following condition

$\epsilon = L - [\lambda\,(z + 1) + q\, FV].$

3.3 Numerical Applications

We used nine orbits with the initial and final position vectors and Δt listed in Table I, components of the position vectors are expressed in geocentric canonical unit ER (ER=6378.1363 km) and time is expressed in unit of time TU (TU = 13.446849 solar minute).

Computational algorithm 2 is then applied for each orbit with $\mu = 1$, Tol $= 10^{-8}$ and the computational check is satisfied within this tolerance, the solutions of the basic equation of the universal Lambert's problem and the check ε are listed for each case in Table II.

Table I. Initial and final position vectors of the test orbits

Orbits	x_1	y_1	z_1
O_1	0.702533	0.380583	-0.299824
O_2	4.54359	-4.52863	-5.66845
O_3	-0.483842	7.37763	1.49529
O_4	0.303061	-0.168102	0.666923
O_5	-0.0173445	-0.333483	-3.42012
O_6	-4.70915	3.25104	3.52456
O_7	1.72008	0.225063	0.674954
O_8	-0.323474	-1.47143	-3.81656
O_9	1.9652	0.72978	1.96275

Orbits	x_2	y_2	z_2	Δt
O_1	0.702533	0.380583	-0.299824	1.22361
O_2	4.54359	-4.52863	-5.66845	2.90691
O_3	-0.483842	7.37763	1.49529	10.
O_4	0.303061	-0.168102	0.666923	1.5
O_5	-0.0173445	-0.333483	-3.42012	2.28939
O_6	-4.70915	3.25104	3.52456	8.
O_7	1.72008	0.225063	0.674954	2.231
O_8	-0.323474	-1.47143	-3.81656	1.20888
O_9	1.9652	0.72978	1.96275	2.231

Table II. The solution s of the universal Lambert's problem for each test orbits

	s	Check
Orbit 1 elliptic	1.2 4687	-2.22045×10^{-16}
Orbit 2 hyperbolic	-0.66667	0.
Orbit 3 parabolic	0.	8.88178×10^{-16}
Orbit 4 elliptic	1.15666	1.33227×10^{-15}
Orbit 5 hyperbolic	-1.24987	0.
Orbit 6 parabolic	0.	0.
Orbit 7 elliptic	1.02598	4.44089×10^{-16}
Orbit 8 hyperbolic	-0.999998	-1.11022×10^{-16}
Orbit 9 parabolic	0.	0.

Conclusion

In concluding the present paper, an iterative method of arbitrary positive integer order of convergence ≥ 2 is developed for solving the basic equation of universal Lambert's problem. The method is characterized by: (1) it is of dynamic nature in the sense that, on going from one iterative scheme to the subsequence one, only additional instruction is needed, (2) it does not need any priori knowledge of the initial guess, a property which avoids it from falling in the critical situations between divergent to very slow convergent solution, that may exist in other numerical methods depending on initial guess as we mentioned before.

The algorithm based on powerful technique for solving transcendental equation without any priori knowledge of the initial guess, this technique is known as homotopy continuation method.

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