Kalman Filter for Fractional Order Singular Systems

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Abstract: In this paper, a Kalman filter for fractional order singular systems has been proposed. To design the Kalman filter, the fractional order singular system is decomposed into two sub-systems by using several transformations. The first sub-system is a dynamic system with fractional order non-singular state equations and the second sub-system is a static system in which the output is a combination of fractional derivatives of its input. Then, the Kalman filter has been elaborated for the system with an index 1 and 2. In the case of index 2, the identification method should be used due to the existence of color noise. At the end, two examples are used to demonstrate efficiency of the proposed method.

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1. Introduction

The history of fractional calculus started almost at the same time when classical calculus was established [1]. In recent years, fractional systems (including fractional derivative or integral) have been the concern of many researchers due to the wide range of their applications[2]. The modelling of physical phenomena such as heat conduction [3], dielectric polarization[4], electromagnetic waves[5] and diffusion waves[6] are examples of fractional systems. Furthermore, fractional order controllers such as the fractional PID controller [7-8] have already been implemented to improve the performance of closed loop control systems.

On the other hand, singular systems which are a combination of algebraic and dynamic equations- could be found in many applications, such as robotic, electrical networks and biomedical engineering systems[9]. These types of systems have complexities and special theorems that complicate their problems in comparison with nonsingular systems. In the field of fractional order singular systems, very few studies exist. So, there are many challenging and unsolved problems.

Due to the wide range application of state variable estimation for stochastic systems, the design of the Kalman filter for fractional order singular systems will be important. The Kalman filter problem for discrete fractional order systems has been solved in [10]. Also, the Kalman filter design for singular systems has been studied in [11-17].

In this paper, a Kalman filter for fractional order singular systems with an index of 1 and 2 is designed. The paper is organized as follows. At first, singular systems are introduced. Then, fractional order singular systems are presented in section 3. The Kalman filter for fractional order singular systems is considered in section 4. Section 5 includes two examples to verify the effectiveness of the proposed method. The paper is concluded in section 6.

2. Introduction of Singular System

Consider a singular system described by

$$\begin{aligned} E\dot{x}(t) &= Jx(t) + K_1u(t) + K_2v_1(t) \\ y(t) &= Lx(t) + v_2(t) \end{aligned} \tag{1}$$

Where $x(t)\in R^{n\times 1}$, $u(t)\in R^{m\times 1}$, $v_1(t)\in R^{p\times 1}$ and $v_2(t)\in R^{p\times 1}$ are state vector, input vector, system noise and measurement noise, respectively. E is a square singular matrix with $n\times n$ _dimension and the other matrices have appropriate dimensions. System noise and measurement noise are both white and uncorrelated with zero mean and the covariance matrices Q' and R .

In the literature, singular systems are also called differential-algebraic systems [18], generalized state space systems [19], descriptor systems [20] and semi- state systems [21].

One of the main properties of singular systems is derivative of the input vector effect in the system state variable. Another property is that the initial condition of singular systems can not be an arbitrary value. In other words, for initial condition x_0 , there may not exist a solution or a solution may not be unique. A different property of singular systems in comparison with non-singular systems is that the transfer function of the system may not be strictly proper.

Theorem 1[22]- Singular system (1) is regular if and only if there exists a scalar λ such that $(\lambda E - J)^{-1}$ exists.■

In solving a singular problem, assuming regularity of the system is necessary to ensure existence and uniqueness of the solution.

Definition 1[22]- the index of $\hat{E}$ is the least non-negative k which satisfies (2):

$$\operatorname{Rank}\left(E^{k}\right) = \operatorname{Rank}\left(E^{k+1}\right) \tag{2}$$

The index of singular system (1) is equal to the index of the $\hat{E}$ matrix , which is defined by (3).

$$\hat{E} = (\lambda E - J)^{-1} E \tag{3}$$

3. Introduction of fractional order singular system

A fractional order singular system is described by the following equations:

$$E\,D^{\alpha} x(t) = Jx(t) + K_1 u(t) + K_2 v_1(t)$$

$$y(t) = Lx(t) + v_2(t) \tag{4}$$

Where D^{α} is the Riemann-Liouville fractional derivative defined by the following equation, for $0 < \alpha < 1$.

$$D^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau \tag{5}$$

also $v_1(t) \in R^{p\times 1}$ and $v_2(t) \in R^{p\times 1}$ are system noise and measurement noise which are both white and uncorrelated with zero mean and the covariance matrices Q' and R , respectively.

Theorem 2[23]- Fractional order singular system (4) is regular if and only if there exist a scalar λ such that $\left(\lambda^{\alpha} E - J\right)^{-1}$ exists.■

Definition 2- The index of fractional order singular system is defined similar to the index of the singular system and equal to the index of $\hat{E}$ which is defined as the following:

$$\hat{E} = (\lambda^{\alpha} E - J)^{-1} E$$

Theorem 3- Consider the fractional singular system (4). If (4) is regular, then its solution can be described as:

$$D^{\alpha} s_1(t) = As_1(t) + B_1 u(t) + B_2 v_1(t) \tag{6}$$

$$s_2(t) = -D_1 u(t) - D_2 v_1(t) - \sum_{i=1}^{m-1} N^i D_1 (\underbrace{D^{\alpha} D^{\alpha} \ldots D^{\alpha}}_{i} u(t))$$

$$- \sum_{i=1}^{m-1} N^i D_2 (\underbrace{D^{\alpha} D^{\alpha} \ldots D^{\alpha}}_{i} v_1(t)) \tag{7}$$

$$y(t) = LQ \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} + v_2(t) + B_2 v_1(t)$$

$$= C_1 s_1(t) + C_2 s_2(t) + v_2(t) + B_2 v_1(t) \tag{8}$$

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = Q^{-1} x(t)$$

Where P and Q are non-singular matrices such that the transformation

$$PEQQ^{-1} D^{\alpha} x(t) = PJQQ^{-1} x(t) + P\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} u(t) \\ v_1(t) \end{bmatrix} \tag{9}$$

gives the system

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} Q^{-1} D^{\alpha} x(t) = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} Q^{-1} x(t) + \begin{bmatrix} B_1 & B_2 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} u(t) \\ v_1(t) \end{bmatrix} \tag{10}$$

where N is a nilpotent matrix.

Proof:

Consider (4). If the system is regular, according to the singular theorem, there exist transformation matrices P and Q such that the decomposed fractional singular system is (10). By defining a new state variable,

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = Q^{-1} x(t) \tag{11}$$

system (10) can be written as

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} D^{\alpha} s_1(t) \\ D^{\alpha} s_2(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} B_1 & B_2 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} u(t) \\ v_1(t) \end{bmatrix} \tag{12}$$

$$y(t) = LQ \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} + v_2(t) \tag{13}$$

Now, if $N = 0$, it can be stated that

$$s_2 = -D_1 u(t) - D_2 v_1(t) \tag{14}$$

and the proof is made.

If $N \neq 0$ we can multiply the second row of (12) with N to get

$$N^2 D^{\alpha} s_2(t) = Ns_2(t) + ND_1 u(t) + ND_2 v_1(t) \tag{15}$$

We now fractional differentiate (15) and insert the second row of (12). This gives

$$s_2(t) = -D_1 u(t) - D_2 v_1(t)$$

$$- ND_1 D^{\alpha} u(t) - ND_2 D^{\alpha} v_1(t) + N^2 D^{\alpha} D^{\alpha} s_2(t)$$

If $N^2 = 0$ the proof is made, otherwise we just continue until $N^m = 0$. We would then arrive at an expression like

$$
\begin{aligned}
s_2(t) = {} & -D_1u(t) - D_2v_1(t) \\
& -\sum_{i=1}^{m-1} N^i D_1 (\underbrace{D^\alpha D^\alpha ... D^\alpha}_{i} u(t)) \\
& -\sum_{i=1}^{m-1} N^i D_2 (\underbrace{D^\alpha D^\alpha ... D^\alpha}_{i} v_1(t))
\end{aligned}
$$

And the proof is complete.■

4. Kalman filter for fractional order singular system

In [15], to design the Kalman filter for a singular system, the covariance matrix is initially considered as $P(t) = E\left[x(t)x^T(t)\right]$. Then by differentiation from both the sides of the equation and the use of system equation, a riccati equation is obtained for the system.

This approach has a basic problem for a fractional singular system. The reason is that in fractional calculus, the derivative of the product of two functions is as follows:

$$D^\alpha\left(x(t)x^T(t)\right) = \sum_{k=0}^{\infty}\binom{\alpha}{k} D^{\alpha-k}(x(t))D^k(x(t)) \tag{16}$$

In other words, if we want to constitute a covariance matrix and derivate it, we will be faced with a series of infinite terms. Also, only a fractional derivative with α order of state variable is obtained from the system equation. While in this series, we also need another fractional derivative of the state variable.

In this section, the Kalman filter for fractional order singular systems is designed. First, the fractional singular system is decomposed into two static and dynamic subsystems. Then, the Kalman filter is designed for each subsystem. In order to solve this problem, two cases of index 1 and 2 are considered for the fractional order singular system.

Theorem 4- The Kalman filter for the fractional singular system (4) with an index of 1 is given by the following set of equations:

$$J' = U_1R_1^{-1} \tag{17}$$

$$
\begin{aligned}
\hat{s}_1(k+1|k) = {} & (A - J'C_1)s_1(k|k) + Bu(k) + J'y(k) \\
& - J'C_2D_1u(k) - \sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}\hat{s}_1(k+1-j|k+1-j)
\end{aligned}
$$

(18)

$$
\begin{aligned}
P_1(k+1|k) = {} & \left(A - J'C_1 + \binom{\alpha}{1}I_n\right)P_1(k|k)\left(A - J'C_1 + \binom{\alpha}{1}I_n\right)^T \\
& + Q_1 + \sum_{j=2}^{k+1}\binom{\alpha}{j}P_1(k+1-j|k+1-j)\binom{\alpha}{j}
\end{aligned}
$$

$$K_k = P_1(k|k-1)C_1^T\left(C_1P_1(k|k)C_1^T + R_1\right)^{-1} \tag{19}$$

(20)

$$\hat{s}_1(k|k) = s_1(k|k-1) + K_k(y(k) - C_1\hat{s}_1(k|k-1) + C_2D_1u(k))$$

(21)

$$P_1(k|k) = \left(I - K_kC_1\right)P_1(k|k-1)$$

$$\hat{s}_2(k|k) = -D_1u(k) \tag{22}$$

(23)

$$P_2(k|k) = E\left[\left(s_2(k) - \hat{s}_2(k)\right)\left(s_2(k) - s_2(k)\right)^T\right] = D_2Q'D_2^T$$

(24)

$$\hat{x}(k|k) = Qs(k|k) \Rightarrow \begin{bmatrix}\hat{x}_1(k|k)\\ \hat{x}_2(k|k)\end{bmatrix} = Q\begin{bmatrix}s_1(k|k)\\ s_2(k|k)\end{bmatrix} \tag{25}$$

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha-j+1)} \tag{26}$$

Where Γ is the Gamma function which is a generalization of the factorial concept for non-integer numbers and $\hat{s}_i(k|k)$ is the estimation of $s_i(k)$ and is defined as:

$$\hat{s}_i(k|k) = E\left[s_i(k)|y(1),...,y(k)\right] \tag{27}$$

Also, $P_1(k|k)$ and $P_2(k|k)$ are the estimation error covariance matrices of the state of first and second subsystems, respectively.

Proof:

By assuming 1 for the index of the system, the solution of the system is obtained as follows:

$$D^\alpha s_1(t) = As_1(t) + B_1u(t) + B_2v_1(t) \tag{28}$$

$$s_2(t) = -D_1u(t) - D_2v_1(t)$$

$$\begin{bmatrix}s_1(t)\\ s_2(t)\end{bmatrix} = Q^{-1}x(t) \tag{29}$$

$$y(t) = LQ\begin{bmatrix}s_1(t)\\ s_2(t)\end{bmatrix} + v_2(t) \tag{30}$$

By considering

$$LQ = \begin{bmatrix}C_1 & C_2\end{bmatrix}$$

And from (28),

$$
\begin{aligned}
y(t) &= C_1s_1(t) + C_2s_2(t) + v_2(t) \\
&= C_1s_1(t) - C_2D_1u(t) - C_2D_2v_1(t) + v_2(t)
\end{aligned} \tag{31}
$$

As can be observed, a continuous Kalman filter for the first subsystem should be designed. However, in a continuous fractional system, unlike an integer system, a dynamic equation for the estimation error covariance matrix cannot be found as a result of the generalized Leibniz rule (14). Therefore, in the procedure of designing a Kalman filter for a fractional system similar to an integer one, a problem is encountered in the step of covariance matrix. Thus, for the estimation of s_1, the first subsystem and output equation are sampled(sampling time is 1).

$$\Delta^{\alpha} s_1(k+1) = A s_1(k) + B_1 u(k) + v_1'(k) \tag{32}$$

$$s_1(k+1) = \Delta^{\alpha} s_1(k+1) - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} s_1(k+1-j) \tag{33}$$

$$y(k) = C_1 s_1(k) - C_2 D_1 u(t) + v_2'(k) \tag{34}$$

For simplicity, by defining

$$\begin{aligned} v_1'(k) &= B_2 v_1(k) \\ v_2'(k) &= -C_2 D_2 v_1(k) + v_2(k) \end{aligned} \tag{35}$$

their covariance matrices are obtained as follows:

$$\begin{aligned} E\left[v_1'(k) v_1'(j)^T\right] &= B_2 Q' B_2^T \delta_{kj} = Q_1 \delta_{kj} \\ E\left[v_2'(k) v_2'(j)^T\right] &= C_2 D_2 Q' D_2^T C_2^T \delta_{kj} = R_1 \delta_{kj} \\ E\left[v_1'(k) v_2'(j)^T\right] &= -B_2 Q' D_2^T C_2^T \delta_{kj} = U_1 \delta_{kj} \end{aligned} \tag{36}$$

In [10], the fractional Kalman filter has been designed for uncorrelated system noise and measurement noise. However, as can be seen from (36), these noises are correlated. Therefore, it is necessary to change the Kalman filter equations presented in [10].

The approach to solving this problem is to reformulate the fractional dynamic equation and cause the noise doses not to correlate with each other. Subsequently, the fractional Kalman filter presented in [10] is used to design a Kalman filter for the first subsystem. This method is derived in detail as shown below.

An item which consists of output equation and equals zero is added to the right hand of (34). The equation is

$$\begin{aligned} s_1(k+1) &= A s_1(k) + B_1 u(k) + v_1'(k) \\ &\quad - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} s_1(k+1-j) \\ &\quad + J'\left(y(k) - C_1 s_1(k) + C_2 D_1 u(t) - v_2'(k)\right) \\ &= \left(A - J' C_1\right) s_1(k) + B_1 u(k) \\ &\quad - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} s_1(k+1-j) \\ &\quad + J'\left(y(k) + C_2 D_1 u(t)\right) + v_1'(k) - J' v_2'(k) \end{aligned} \tag{37}$$

Where J' is a coefficient matrix to be determined. New system noise $v_1^*(k)$ is defined as

$$v_1^*(k) = v_1'(k) - J' v_2'(k) \tag{38}$$

The expectation of $v_1^*(k)$ is calculated by

$$E\left[v_1^*(k)\right] = E\left[v_1'(k)\right] - J' E\left[v_2'(k)\right] = 0 \tag{39}$$

The covariance between system noise $v_1^*(k)$ in (38) and measurement noise $v_2'(k)$ is calculated by

$$E\left[\left(v_1'(k) - J' v_2'(k)\right) v_2'(j)^T\right] = U_1 \delta_{kj} - J' R_1 \delta_{kj} \tag{40}$$

The coefficient matrix J' is determined by $U_1 - J' R_1 = 0$. By rearranging the equation, we obtain

$$J' = U_1 R_1^{-1} \tag{41}$$

That is to say, the Kalman filter presented in [10] can be used to design a Kalman filter for the first subsystem. By some manipulations, (18) - (22) will be obtained. It is noted that equation (19) is the main difference between the Kalman filter for the fractional system with an integer order Kalman filter. The calculation of the covariance error matrix ($P_1(k+1|k)$) depends on the values of the covariance matrices in previous time samples (i.e. $P_1(k-1|k-1), P_1(k-2|k-2), \ldots$), in addition to $P_1(k-1|k-1)$. The other difference between the proposed Kalman filter and the conventional Kalman filter is the insertion of $C_2 D_1 u(k)$ to the state estimation equation.

Now, s_2 is estimated. As shown in (29), s_2 is an algebraic equation including the deterministic input and system noise. So, the best estimation is the mean of the algebraic equation.

$$\hat{s}_2(k|k) = -D_1 u(k) \tag{42}$$

The estimation error covariance matrix can be calculated by (24). The initial condition of (24) is:

$$P_{2_0}^{\ -} = \begin{bmatrix} 0 & I \end{bmatrix} Q^{-1} E\left[x_0 x_0^T\right] Q^{T-1} \begin{bmatrix} 0 & I \end{bmatrix}^T \tag{43}$$

$$P_{2_0}^{\ +} = D_2 Q' D_2^T \tag{44}$$

By looking at (43)-(44), the covariance matrix P_2 may have a discontinuity at zero time. This is another important difference between singular and non singular systems.

Also, with regard to (28)-(29), one can see that the system noise is inserted to s_1 and s_2, simultaneously. So it is necessary to calculate $P_{12}(k)$. However, if B_2 or D_2 are zero matrices, then they will not need to be calculated. By some manipulation, we have:

$$\begin{aligned} P_{12}(k) &= E\left[\left(s_1(k) - \hat{s}_1(k)\right)\left(s_2(k) - s_2(k)\right)^T\right] \\ &= E\left[e_1(k) e_2^T(k)\right] = -K_k C_2 D_2 Q' \end{aligned} \tag{45}$$

Now, we present an important lemma which is used for the design of the Kalman filter for fractional singular systems with an index of 2.

Lemma 1- Assume that v_1 and v_2 are two uncorrelated zero mean white noises. Their covariance matrices are Q' and R, respectively. Then v_3 described by (46) is a color noise.

$$v_3(k) = Av_1(k) - B\Delta^{\alpha}v_1(k) + v_2(k) \tag{46}$$

Proof:

The covariance of $v_3(k)$ is calculated by

$$\begin{aligned}
&E\left[v_3(k)v_3^T(j)\right] \\
&= A\,E\left[v_1(k)v_1^T(j)\right]A^T + E\left[v_2(k)v_1^T(j)\right]A^T \\
&\quad - A\,E\left[v_1(k)\Delta^{\alpha}v_1^T(j)\right]B^T + A\,E\left[v_1(k)v_2^T(j)\right] \\
&\quad - B\,E\left[\Delta^{\alpha}v_1(k)v_1^T(j)\right]A^T - BE\left[\Delta^{\alpha}v_1(k)v_2^T(j)\right] \\
&\quad + B\,E\left[\Delta^{\alpha}v_1(k)\Delta^{\alpha}v_1^T(j)\right]B^T - E\left[v_2(k)\Delta^{\alpha}v_1^T(j)\right]B^T \\
&\quad + E\left[v_2(k)v_2^T(j)\right]
\end{aligned}$$

The above equation is simplified as

$$\begin{aligned}
&E\left[v_3(k)v_3^T(j)\right] \\
&= A\,E\left[v_1(k)v_1^T(j)\right]A^T - A\,E\left[v_1(k)\Delta^{\alpha}v_1^T(j)\right]B^T \\
&\quad - B\,E\left[\Delta^{\alpha}v_1(k)v_1^T(j)\right]A^T + B\,E\left[\Delta^{\alpha}v_1(k)\Delta^{\alpha}v_1^T(j)\right]B^T \\
&\quad + E\left[v_2(k)v_2^T(j)\right] \\
&= AQ'A^T\,\delta_{kj} - A\sum_{l=0}^{j}(-1)^l\binom{\alpha}{l}Q'\,\delta_{k,j-l}B^T \\
&\quad - B\sum_{l=0}^{k}(-1)^l\binom{\alpha}{l}Q'\,\delta_{k,j-l}A^T \\
&\quad + B\sum_{l=0}^{k}\sum_{m=0}^{j}\binom{\alpha}{l}\binom{\alpha}{m}Q'\delta_{k-l,j-m}B^T + R
\end{aligned}$$

It is clear from the above equation that the covariance matrix is nonzero for $k \neq j$. Therefore, v_3 is a color noise.■

In the next theorem, we design the Kalman filter for the fractional singular system with an index of 2.

Theorem 5- The Kalman filter for fractional singular system (4) with an index of 2 is given by the following set of equations:

$$J' = U_2R_2^{-1} \tag{47}$$

$$\begin{aligned}
\hat{\chi}_1(k+1|k) &= (\bar{A} - J'\bar{C})\chi_1(k|k) + \bar{B}u_k + J'y(k) \\
&\quad - J'C_2D_1u(k) - J'C_2D_1\Delta^{\alpha}u(k) \\
&\quad - \sum_{j=1}^{k+1}\Gamma_j\,\hat{\chi}_1(k+1-j|k+1-j)
\end{aligned} \tag{48}$$

$$\begin{aligned}
\bar{P}_1(k+1|k) &= \left(\bar{A} - J'\bar{C} + \Gamma_1\right)\bar{P}_1(k|k)\left(\bar{A} - J'\bar{C} + \Gamma_1\right)^T + Q_2 \\
&\quad + \sum_{j=2}^{k+1}\Gamma_j\bar{P}_1(k+1-j|k+1-j)\Gamma_j
\end{aligned} \tag{49}$$

$$K_k = \bar{P}_1(k|k-1)\bar{C}^T\left(\bar{C}\,\bar{P}_1(k|k-1)\bar{C}^T + R_2\right)^{-1} \tag{50}$$

$$\begin{aligned}
\hat{\chi}_1(k|k) &= \chi_1(k|k-1) \\
&\quad + K_k(y(k) - C_1\hat{\chi}_1(k|k-1) + C_2D_1u(k) + C_2ND_1\Delta^{\alpha}u(k))
\end{aligned} \tag{51}$$

$$P_1(k|k) = \left(I - K_k\bar{C}\right)P_1(k|k-1) \tag{52}$$

$$\hat{\chi}_1(k|k) = \begin{bmatrix}\hat{s}_1(k|k) \\ \hat{\eta}(k|k)\end{bmatrix} \tag{53}$$

$$\hat{s}_2(k|k) = -D_1u(k) - ND_1\Delta^{\alpha}u(k) \tag{54}$$

$$\begin{aligned}
P_2(k|k) &= D_2Q'D_2^T + D_2Q'D_2^TN^T + ND_2Q'D_2^T \\
&\quad + ND_2\left(\sum_{j=0}^{k}\binom{\alpha}{j}Q'\binom{\alpha}{j}\right)D_2^TN^T
\end{aligned} \tag{55}$$

$$\hat{x}(k|k) = Qs(k|k) \Rightarrow \begin{bmatrix}\hat{x}_1(k|k) \\ \hat{x}_2(k|k)\end{bmatrix} = Q\begin{bmatrix}s_1(k|k) \\ s_2(k|k)\end{bmatrix} \tag{56}$$

Where $\hat{s}_i(k|k), i = 1,2$, $P_1(k|k)$ and $P_2(k|k)$ are defined as theorem 4.

Proof:

If the index of the system is 2, then its solution is obtained as follows (sampling time is 1):

$$\begin{aligned}
&\Delta^{\alpha}s_1(k+1) = As_1(k) + B_1u(k) + v_1'(k) \\
&s_1(k+1) = \Delta^{\alpha}s_1(k+1) - \sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}s_1(k+1-j)
\end{aligned} \tag{57}$$

$$\begin{aligned}
s_2(k) &= -D_1u(k) - D_2v_1(k) - ND_1\Delta^{\alpha}u(k) \\
&\quad - ND_2\Delta^{\alpha}v_1(k)
\end{aligned} \tag{58}$$

$$y(k) = C_1s_1(k) - C_2D_1u(k) - C_2ND_2\Delta^{\alpha}u(k) + v_2'(k) \tag{59}$$

Where

$$\begin{aligned}
&v_1'(k) = B_2v_1(k) \\
&v_2'(k) = -C_2D_2v_1(k) - C_2ND_2\Delta^{\alpha}v_1(k) + v_2(k)
\end{aligned} \tag{60}$$

By lemma 1, it can be seen that $v_2'(k)$ is a color noise. In order to design the Kalman filter, we find a dynamic system which its input is white noise and its output is color noise with a mean and covariance matrix similar to $v_2'(k)$ by means of identification methods.

$$\begin{aligned} \eta(k+1) &= \xi\eta(k)+\psi w(k) \\ v_2'(k) &= \varphi\eta(k)+\phi w(k) \end{aligned} \tag{61}$$

Now, the above system and the first subsystem are augmented. Then the Kalman filter is designed for the new system and (48)-(52) are obtained. Similar to theorem 4, the second subsystem is a static system, so the best estimation of the second state is the mean of the algebraic equation and (54)-(55) is derived.■

5. Numerical Examples

A. Consider the fractional order singular system with an index of 1, described by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D^{0.5}x_1(t) \\ D^{0.5}x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0.7071 \\ 1 \end{bmatrix}u(t) + \begin{bmatrix} 0.7071 \\ 0 \end{bmatrix}v_1(t) \tag{62}$$

$$y(t) = \begin{bmatrix} 0 & 1.4142 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v_2(t) \tag{63}$$

Where $v_1(t)$ and $v_2(t)$ are white noise with zero mean value and the covariance matrices as:

$$E\left[v_1v_1^T\right]=0.1 \quad , \quad E\left[v_2v_2^T\right]=0.1 \tag{64}$$

The above system is regular for $\lambda=0$. $\hat{E}$ is obtained as:

$$\hat{E}=(\lambda E-J)^{-1}E=-J^{-1}E=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{65}$$

We have

$$rank(\hat{E})=1 \quad , rank\left(E^2\right)=1$$

So, the index of this system is one. According to the singular theorem, there exist matrices P and Q

$$P=\begin{bmatrix} 1.4142 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad Q=\begin{bmatrix} 0.7071 & 1 \\ 0.7071 & 0 \end{bmatrix} \tag{66}$$

Such that the system is decomposed as follows:

$$\begin{aligned} D^{0.5}s_1 &= -s_1(t)+u(t)+0.1v_1(t) \\ s_2(t) &= u(t) \\ y(t) &= s_1(t)+v_2(t) \end{aligned} \tag{67}$$

Figure 1 shows the first subsystem state variable and its estimation. It is clear that the state variable has been estimated very well.

For the second subsystem state variable, we have

$$s_2(t)=u(t) \tag{68}$$

In this case, the estimation of the state is coincided by the state exactly, because this state variable is independent of noise and is equal to the deterministic input.

Figures (2)-(3) show the state variable of system (64) and their estimations. It is evident that the proposed method has an appropriate efficiency.

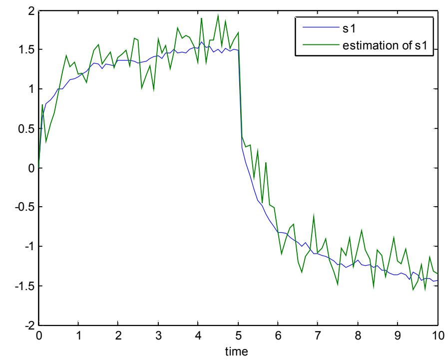

Figure 1. first subsystem state variable and its estimation

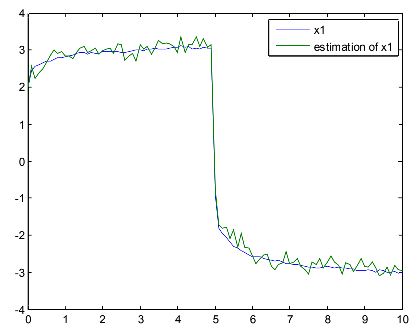

Figure 2. first state variable and its estimation

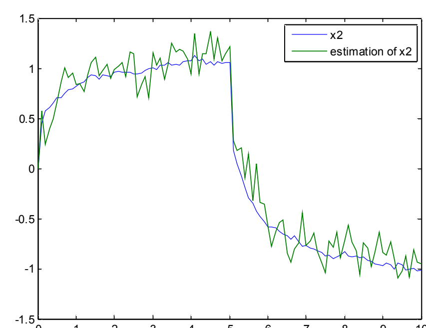

Figure 3. second state variable and its estimation

B. Consider the fractional order singular system (69).

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}D^{0.5}x_1(t)\\D^{0.5}x_2(t)\\D^{0.5}x_3(t)\end{bmatrix}=\begin{bmatrix}0&-1&0.5\\0&0&-0.5\\-1&0&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}+\begin{bmatrix}0\\0.45\\0\end{bmatrix}u(t)+\begin{bmatrix}-1.2\\2.6\\-1\end{bmatrix}v_1(t) \quad (69)$$

$$y(t)=\begin{bmatrix}2.27&2.27&0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}+v_2(t) \quad (70)$$

Where $v_1(t)$ and $v_2(t)$ are white noise with zero mean value and the covariance matrices as:

$E\left[v_1v_1^T\right]=0.1$, $E\left[v_2v_2^T\right]=0.1$

The above system is regular for $\lambda=0$. $\hat{E}$ is obtained as:

$$\hat{E}=\left(\lambda^{0.5}E-A\right)^{-1}E=-A^{-1}E=\begin{bmatrix}0&0&0\\1&-1&0\\0&-2&0\end{bmatrix}$$

We have

$rank\left(\hat{E}\right)=2$, $rank\left(\hat{E}^2\right)=1$, $rank\left(\hat{E}^3\right)=1$

So, the index of this system is two. Using non-singular matrices P and Q

$$P=\begin{bmatrix}2.236&2.236&2.236\\-0.8944&0&0\\0&0&-1\end{bmatrix}$$

$$Q=\begin{bmatrix}0&0&1\\0.4472&0&-1\\0.8944&-2.236&-2\end{bmatrix}$$

We obtain

$$\begin{bmatrix}1&0&0\\0&0&-0.8944\\0&0&0\end{bmatrix}\begin{bmatrix}D^{0.5}s_1(t)\\D^{0.5}s_{21}(t)\\D^{0.5}s_{22}(t)\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}s_1(t)\\s_{21}(t)\\s_{22}(t)\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}u(t)+\begin{bmatrix}1\\1\\1\end{bmatrix}v_1(t) \quad (71)$$

$$y(t)=s_1(t)+v_2(t) \quad (72)$$

The first subsystem state variable and its estimation are shown in figure (4). For the second subsystem, we have

$$\begin{bmatrix}s_{21}(t)\\s_{22}(t)\end{bmatrix}=\begin{bmatrix}-1\\-1\end{bmatrix}v_1(t)-\begin{bmatrix}-0.8944\\0\end{bmatrix}D^{0.5}v_1(t) \quad (73)$$

$$\begin{bmatrix}\hat{s}_{21}\\\hat{s}_{22}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad (74)$$

Also, the estimation of the states of system (69) are obtained as:

$$\begin{aligned}\hat{x}_1(t)&=s_{22}(t)\\\hat{x}_2(t)&=0.4472s_1(t)-\hat{s}_{22}(t)\\\hat{x}_3(t)&=0.8944s_1(t)-2.236\hat{s}_{21}(t)-2s_{22}(t)\end{aligned} \quad (75)$$

The results are shown in figures (5)-(7). These figures show that the proposed Kalman filter works properly. As can be seen in figure (5), the estimation of the first state variable is zero, because this state is equal to system noise for which the mean is zero. So, its estimation becomes zero.

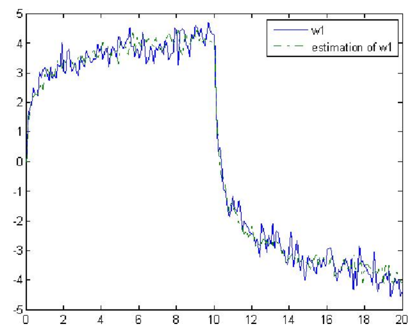

Figure 4. first subsystem state variable and its estimation

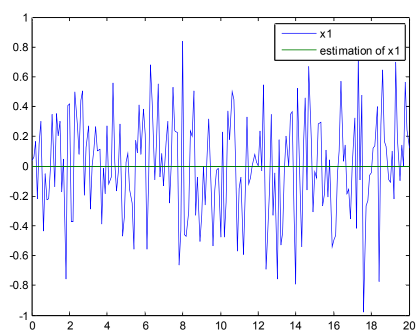

Figure 5. first state variable and its estimation

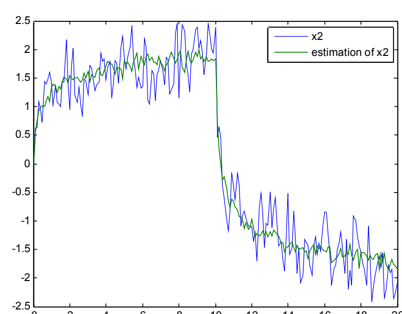

Figure 6. second state variable and its estimation

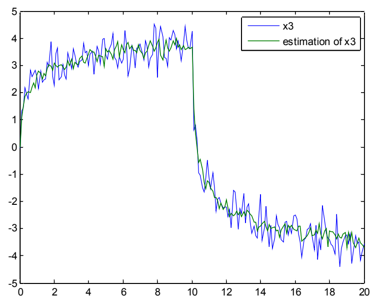

Figure 7. third state variable and its estimation

6. conclusion

In this paper, the design of Kalman filters for a fractional order singular system has been proposed. To approach this goal, first the fractional order singular system has been decomposed into two subsystems and then the Kalman filters are elaborated for each subsystem. Also, the main difference between the fractional singular system and the fractional non-singular system has been expressed in this regard. We showed that in the design of the kalman filter for a system with index of 2, we have a color noise and it is necessary to use an identification method to solve the problem. At the end, the validity of the proposed method has been demonstrated by the simulations.

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