Order Statistics From Discrete Gamma Distribution

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Abstract: This paper introduces the subject of order statistics os and its moments for two kinds of discrete random variables rv's drawn from two parameters discrete gamma distribution or DGD (α, q). The first kind is the independent identically distributed discrete rv's or iid, and the second kind is the independent non-identically distributed inid discrete rv's. A brief look at DGD (α, q) is given including its special cases DGD (2, q), DGD (3, q) and DGD (4, q). For iid and inid discrete rv's equations of distributions of single order statistics and their moments used here are presented and its applications for DGD (2, q) are studied. Distribution of the joint os and the probability mass function pmf of the range from DGD (2,q) is obtained for the iid case. The k^{th} moments of single os for inid rv's following DGD(2,q) is obtained and the mean of the largest and smallest os for them are also calculated for n=3.

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1- Introduction

Random variables can be divided into three categories: continuous, discontinuous and discrete. Any kind of these random variables can be either identically or non-identically distributed. The observations of x_i can be either independent or dependent.

If $x_1, x_2, \ldots, x_n$ is a random sample from a distribution and if the $x_i's$, $i = 1,2, \ldots, n$, are arranged in ascending order of magnitude, then it is written as $x_1 \leqq x_2 \leqq x_3 \ldots \leqq x_n$, and x_i, $i = 1,2, \ldots, n$, is called the ith order statistic of the random sample $x_1, x_2, \ldots, x_n$.

The subject of os have in recent times come to play an important role in statistical inference partly because some of their properties do not depend upon the distribution from which the random sample is obtained. Earlier studied of os were concentrated on distributions of os from iid continuous rv's. A researcher can find hundreds- if not thousands of articles related to this branch. Then studies on os from iid discrete rv's appeared in the literature. But not many articles related to this subject can be found in the literature compared to those of the continuous rv's. The reason of this is in my opinion is due to the complicity of the mathematical formulas of the pmf and cdf of os for the discrete rv's (Arnold, B.C. , Balakrishnan, N. and Nagarag, H.N. 1992) (Abdel-Aty 1954) (Siotani 1956).

Then latter on abandon of the (identically-condition) took place in the literature and new studies appeared related to os for independent but not identically distributed inid continuous rv's. The only book written by (Afify 2010) for only inid continuous rv's.

In practice one never samples from a continuous probability distribution (Holland 1975). The failure data are measured as discrete variables such as the number of runs, cycles, or shocks. (Khan, A. , Khaique and Aboummah, A. 1989). Almost always the observed values are actually discrete because they are measured to only a finite number of decimal places and cannot really constitute all points in a continuum (Chakraborty, S. & Chakravarty, D. 2012).

In this paper we will study os from discrete gamma distribution DGD (α,q) for both conditions when the X's are iid and inid discrete rv's.

2-Discrete Gamma Distribution

The pdf of the usual continuous two parameter gamma distribution is given by:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad ; \quad x \geq 0, \alpha > 0, \beta > 0 \qquad (1)$$

The discrete gamma distribution is defined based on discretizing X and α and considering the substitution $q = e^{-\beta}$ (Alhazzani, N.S 2012). The discrete non-negative random variable X is said to have a discrete gamma distribution with parameters (α, q) denoted by DGD (α, q), if its probability mass function pmf is given by:

$$p(\alpha, q) = \Pr(X = x) = C\, p^\alpha x^{\alpha-1} q^x \quad , \; X = 0, 1, 2, \ldots, \qquad (2)$$

where $\alpha = 1, 2, \ldots$ is the shape parameter,

$0 < q < 1$, $p + q = 1$, and C=1 if α=1, and

$$C = \frac{1}{q \sum_{i=1}^{\alpha-2} A(\alpha-1, i) q^i} \text{ , for } \alpha \geq 2 \quad (3)$$

where A (n , m) is known as Eulerian number (Hirzebruch 2008), and is given by:

$$A(n, m) = \sum_{k=0}^{m} (-1)^k \binom{n+1}{k} (m+1-k)^n. \quad (4)$$

The cumulative distribution function cdf of this distribution is given by:

$$P(\alpha, q) = \Pr(X \leq x) = \sum_{w=0}^{x} C\, p^{\alpha}\, w^{\alpha-1} q^{w}, w = 0, 1, 2, ..., \quad (5)$$

The pmf and the cdf of the DGD (α, q) when α = 1 , 2,3 , 4 are respectively:

DGD (1, q)

$$p(1, q) = pq^x \text{ , } x = 0, 1, 2, ... \quad (6)$$

$$P(1, q) = 1 - q^{x+1} \quad (7)$$

DGD (2, q)

$$p(2, q) = x\, p^2 q^{x-1} \text{ , } x = 0, 1, 2, ...$$

$$= x(q^{x-1} - 2q^x + q^{x+1}) \quad (8)$$

$$P(2, q) = 1 - q^x(1+x) + xq^{x+1} \quad (9)$$

DGD (3, q)

$$p(3, q) = \frac{x^2\, p^3\, q^{x-1}}{1+q} \quad (10)$$

$$P(3, q) = \frac{1 + q - (1+x)^2 q^x - (1 - 2x - 2x^2) q^{x+1} - x^2 q^{x+2}}{1+q} \quad (11)$$

DGD (4, q)

$$p(4, q) = \frac{x^3\, p^4\, q^{x-1}}{(1 + 4q + q^2)} \quad (12)$$

$$P(4, q) = \frac{1 + 4q + q^2 - q^x(1 + 3x + 3x^2 + x^3)}{1 + 4q + q^2} + \frac{-q^{x+1}(4 - 6x^2 - 3x^3) - q^{x+2}(1 - 3x + 3x^2 + 3x^3) + x^3 q^{x+3}}{1 + 4q + q^2} \quad (13)$$

3- Distributions of OS for IID RV's Following Discrete Distributions

3.1 Distribution of Single OS From IID RV's Following Discrete Distributions

The formulae of the pmf and cdf of the i^th^ os $\boldsymbol{X_{i:n}}$ used here are given by (Nagaraga 1992):

$$p_{i:n}(x) = C(i:n) \int_{P(x-1)}^{P(x)} w^{i-1} * (1-w)^{n-i}\, dw \quad (14)$$

$$P_{i:n}(x) = C(i:n) \int_{0}^{P(x)} w^{i-1} * (1-w)^{n-i}\, dw \quad (15)$$

Where

$$C(i;n) = i\binom{n}{i} = \frac{n!}{(i-1)!\,(n-i)!} \quad (16)$$

In particular when i=1,n , the pmf and cdf of the smallest $X_{1:n}$ and largest $X_{n:n}$ os are given by:

$$p_{1:n}(x) = C(1:n) \int_{P(x-1)}^{P(x)} (1-w)^{n-1}\, dw = [1 - P(x-1)]^n - [1 - P(x)]^n \quad (17)$$

$$P_{1:n}(x) = n\int_{0}^{P(x)} (1-w)^{n-1}\, dw$$
$$= 1-[1-P(x)]^{n} \qquad (18)$$

$$p_{n:n}(x) = n\int_{P(x-1)}^{P(x)} w^{n-1}\, dw$$

$$= [P(x)]^{n} - [P(x-1)]^{n} \qquad (19)$$

$$P_{n:n}(x) = n\int_{0}^{P(x)} w^{n-1}\, dw$$
$$= [P(x)]^{n} \qquad (20)$$

3.1.1 Distribution of Single OS From IID DGD (α, q) RV's

The pmf of the i[th] os from iid DGD (α, q) distribution is given by:

$$p_{i:n}(x) = C(i:n)\int_{P(x-1)}^{P(x)} w^{i-1} * (1-w)^{n-i}\, dw \qquad (21)$$

Where the upper and lower limits of the integral are given by:

$$P(x) = P_r(X \le x) = \sum_{v=0}^{x} p_{i:n}(v)$$
$$= \sum_{v=0}^{x} C * p^{\alpha} v^{\alpha-1} q^{v} \qquad (22)$$
$$P(x-1) = P_r(X \le x-1) = \sum_{v=0}^{x-1} p_{i:n}(v)$$
$$= \sum_{v=0}^{x-1} C * p^{\alpha} v^{\alpha-1} q^{v} \qquad (23)$$

Where C is as defined in equation 3.

$$p_{i:n}(x) = C(i:n)\int_{\sum_{v=0}^{x-1} C*p^{\alpha}v^{\alpha-1}q^{v}}^{\sum_{v=0}^{x} C*p^{\alpha}v^{\alpha-1}q^{v}} w^{i-1} * (1-w)^{n-i}\, dw \qquad (24)$$

Write w=1-(1-w), then

$$p_{i:n}(x) = C(i:n)\int_{P(x-1)}^{P(x)} [1-(1-w)]^{i-1} (1-w)^{n-i}\, dw \qquad (25)$$

Using binomial expression for the first bracket, we get:

$$p_{i:n}(x) = C(i:n)\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{j}\int_{P(x-1)}^{P(x)} (1-w)^{n+j-i}\, dw \qquad (26)$$

$$p_{i:n}(x) = C(i:n)\sum_{j=0}^{i-1}\frac{\binom{i-1}{j}(-1)^{j+1}}{(n+j-i+1)}$$
$$[(1-P(x))^{n+j-i+1} - (1-P(x-1))^{j+n-i+1}] \qquad (27)$$
$$p_{i:n}(x) = C(i:n)\sum_{j=0}^{i-1}\frac{\binom{i-1}{j}(-1)^{j+1}}{(n+j-i+1)}$$

$$[(1-\sum_{v=0}^{x} C*p^{\alpha}v^{\alpha-1}q^{v})^{n+j-i+1}-(1-\sum_{v=0}^{x-1} C*p^{\alpha}v^{\alpha-1}q^{v})^{j+n-i+1}] \quad (28)$$

Since the expression of the cdf of DGD (α, q) for any α is not given in a compact form, we are going to study the special case of DGD (α, q), where α = 2; i.e. DGD (2,q).

3.1.2 Single OS From IID DGD (2,q) RV's

The pmf of the i^{th} os from iid DGD (2, q) distribution

$$p_{i:n}(x)=C(i:n)\int_{1-q^{x-1}x+(x-1)q^{x}}^{1-q^{x}(1+x)+xq^{x+1}} w^{i-1}*(1-w)^{n-i}\,dw \qquad =C(i;n)\frac{\sum_{j=0}^{i-1}\binom{i-1}{j}*(-1)^{j}}{n+j-i+1}$$
$$[(q^{x-1}x+(x-1)q^{x})^{n+j-i+1}-(q^{x}(1+x)+xq^{x+1})^{n+j-i+1}] \quad (29)$$

The cdf is written as:

$$P_{i:n}(x)=C(i:n)\int_{0}^{1-q^{x}(1+x)+xq^{x+1}} w^{i-1}*(1-w)^{n-i}\,dw$$
$$=C(i;n)\frac{\sum_{j=0}^{i-1}\binom{i-1}{j}*(-1)^{j}}{n+j-i+1}*(1-q^{x}(1+x)+xq^{x+1})^{n+j-i+1} \quad (30)$$

Where $C(i:n)$ is as defined in equation16.

3.1.3 Distribution of the Smallest and Largest OS; $X_{1:n}$ $X_{n:n}$ Following IID DGD (2,q)

The pmf and cdf of the smallest os from iid DGD (2, q) distributions are written as:

$$p_{1:n}(x)=[q^{x-1}\,x-\,q^{x}\,(x-1)]^{n}-\,[q^{x}\,(1+x)-\,q^{x+1}\,x]^{n} \quad (31)$$
$$P_{1:n}(x)\;=\;1-(q^{x}(1+x)-x\,q^{x+1})^{n} \quad (32)$$

The pmf and cdf of the largest os from iid DGD (2, q) distributions are written as:

$$p_{n:n}(x)=(1-\,q^{x}(1+x)+\,q^{x+1}\,x)^{n}-\,(1-q^{x-1}\,x+\,q^{x}(\,x-1))^{n} \quad (33)$$
$$P_{n:n}(x)=(1-\,q^{x}(1+x)+\,q^{x+1}\,x)^{n} \quad (34)$$

3.2 The Joint Distribution of Two OS ; $X_{i:n}$ $X_{j:n}$ From IID DGD (α, q)

The general form of the joint pmf of two os $X_{i:n}$ $X_{j:n}$ used here is given by (Kabe 1969):

$$p_{i,j:n}(x,y)=P_r\{X_{i:n}=x,X_{j:n}=y\}$$
$$=\sum_{k=1}^{i}\sum_{t=1}^{n-j+1}C'\;[-\{P(x)\}^{i-k}\{P(y)-P(x)\}^{j-i+k-1+t}1-\,P(y)+\,\{P(x)\}^{i-k}\{P(y-1)-P(x)\}^{j-i+k-1+t}\{1-P(y-1)\}^{n-j-t+1}+\,\{P(x-1)\}^{i-k}\{P(y)-P(x-1)\}^{j-i+k-1+t}\{1-(1-P(y)\}^{n-j-t+1}-\{P(x-1)\}^{i-k}\{P(y-1)-\,P(x-1)\}^{j-i+k-1+t}\{(1-P(y-1)\}^{n-j-t+1}] \quad (35)$$

Where

$$C'=\frac{n!}{(i-k)!\;(n-j+1-t)!\,(j-i+k-1+t)!} \quad (36)$$

3.2.1 The Joint Distribution of Two OS ; $X_{i:n}$ $X_{j:n}$From IID DGD (2, q)

The joint pmf of two os ; $X_{i:n}$ $X_{j:n}$from iid DGD(2, q)is given by:

$$p_{i,j:n}(x,y)=P_r\{X_{i:n}=x,X_{j:n}=y\}$$
$$=\sum_{k=1}^{i}\sum_{t=1}^{n-j+1}C'\quad[-\{1-q^{x}\,(1+x)+x\,q^{x+1}\,\}^{i-k}\{y\,q^{y+1}-\,q^{y}\,(1+y)+\,q^{x}\,(1+x)-x\,q^{x+1}\}^{j-i+k-1+t}\{q^{y}\,(1+y)-y\,q^{y+1}\,\}^{n-j-t+1}+\{1-q^{x}\,(1+x)+x\,q^{x+1}\}^{i-k}\{(y-1)q^{y}\,-\,q^{y-1}\,y+q^{x}\,(1+x)-x\,q^{x+1}\}^{j-i+k-1+t}\{q^{y-1}\,y-(y-1)\,q^{y}\}^{n-j-t+1}+\;\{1-q^{x-1}\,x+(x-1)q^{x}\}^{i-k}\{y\,q^{y-1}-q^{y}\,(1+y)+\,q^{x-1}\,x-(x-1)q^{x}\}^{j-i+k-1+t}\{q^{y}\,(1+y)-y\,q^{y+1}\}^{n-j-t+1}-\{1-\,q^{x-1}\,x+(x-1)q^{x}\}^{i-k}\{(y-1)q^{y}\,-\,q^{y-1}\,y+\,q^{x-1}\,x-(x-1)q^{x}\}^{j-i+k-1+t}\{q^{y-1}\,y-(y-1)\,q^{y}\}^{n-j-t+1}] \quad (37)$$

3.2.2 The Joint Distribution of the Smallest and Largest $X_{1:n}$ *and* $X_{n:n}$ OS From IID DGD (2,q)

As a special case the joint pmf of $X_{1:n}$ *and* $X_{n:n}$; written as:

where i=1, j=n;

$$p_{1,n:n}(x,y) = -\{y\, q^{y+1} - q^{y}\ (1+y) + q^{x}\ (1+x) - x\, q^{x+1}\}^{n} + \{(y-1)q^{y} - q^{y-1}\ y + q^{x}\ (1+x) - x\, q^{x+1}\}^{n} + \{y\, q^{y+1} - q^{y}(1+y) + q^{x-1}x - (x-1)q^{x}\}^{n} - \{(y-1)q^{y} - q^{y-1}\ y + q^{x-1}x - (x-1)q^{x}\}^{n} \quad (38)$$

3.2.3 The Distribution of the Range From IID DGD (2,q)

The pmf of the range from iid DGD(2,q) is obtained by using the two transformations x = $X_{1:n}$, y = r+ x, $R_{1;n:n} = r$;

Substitute in (38) we get the joint pmf of $X_{1:n}, r$ as:

$$p_{1,n:n}(r,x) = -\{(r+x)\, q^{r+x+1} - q^{r+x}\ (1+r+x) + q^{x}\ (1+x) - x\, q^{x+1}\}^{n} + \{(r+x-1)q^{r+x} - q^{r+x-1}\ (r+x) + q^{x}\ (1+x) - x\, q^{x+1}\}^{n} + \{(r+x)\, q^{r+x+1} - q^{r+x}(1+r+x) + q^{x-1}x - (x-1)q^{x}\}^{n} - \{(r+x-1)q^{r+x} - q^{r+x-1}\ (r+x) + q^{x-1}x - (x-1)q^{x}\}^{n} \quad (39)$$

$$p_{1,n:n}(r,x) = q^{nx}[-\{(r+x)q^{r+1} - q^{r}\ (1+r+x) + \ (1+x) - x\, q\}^{n} + \{(r+x-1)q^{r} - q^{r-1}\ (r+x) + \ (1+x) - x\, q\}^{n} + \{(r+x)q^{r+1} - q^{r}(1+r+x) + q^{-1}x - (x-1)\}^{n} - \{(r+x-1)q^{r} - q^{r-1}\ (r+x) + q^{-1}x - (x-1)\}^{n}] \quad (40)$$

Summing over x we get the marginal pmf of the range r as:

$$p(r) = \sum_{x=0}^{\infty} q^{nx}[-\{(r+x)q^{r+1} - q^{r}\ (1+r+x) + \ (1+x) - x\, q\}^{n} + \{(r+x-1)q^{r} - q^{r-1}\ (r+x) + \ (1+x) - x\, q\}^{n} + \{(r+x)q^{r+1} - q^{r}(1+r+x) + q^{-1}x - (x-1)\}^{n} - \{(r+x-1)q^{r} - q^{r-1}\ (r+x) + q^{-1}x - (x-1)\}^{n}] \quad (41)$$

4- Distributions of OS for INID RV's Following Discrete Distribution

4.1 Distribution of Single OS From INID RV's Following Discrete Distributions

Let X be the discrete variable which may take the values 0,1,2,… with probabilities $p(0), p(1), p(2), \ldots$ respectively, where $p(\alpha) \geqq 0$ and $\sum_{\alpha=0}^{\infty} p(\alpha) = 1$. When x takes only the finite number of values 0,1,2,…,M, we interpret that $p(M+\alpha) = 0$ for $\alpha = 1,2, \ldots$.

Let X_r be the r^th^ os given by $X_1 \le X_2 \le X_3 \le \cdots \le X_n$ where they are arranged from $X_{(i)'s}$ independent non-identically distributed discrete rv with pmf's $p_i(x)$, and cdf $P_i(x)$. Argument involving multinomial trials was used to obtain the pmf of $X_{r.n}$, with the chance of ties is nonzero.

The pmf of the r^th^ os $X_{r.n}(r = 1,2, \ldots, n)$ for inid rv's is given by:

$$P_{r:n}(x) = \sum_{k=0}^{r-1} \sum_{m=0}^{n-r} \frac{\sum_P \prod_{a=1}^{r-k-1} P_{i_a}(x-1) \prod_{b=r-k}^{k+m+1} p_{i_b}(x) \prod_{c=k+m+2}^{n} \left(1 - P_{i_c}(x)\right)}{(r-k-1)!\,(k+m+1)!\,(n-r-m)!} \quad (42)$$

Where $\sum_P$ denotes the sum over all n! permutation $(i_1, i_2, \ldots, i_n)$ of (1,2,…,n) , and $P\ (X_r - 1) = 0$ when $x_i = 0$.

As pointed out by (Vaughan,R.J. and Venables 1972), the pmf of $X_{r:n}$ in eq.42 can be expressed in terms of a permanent as follows:

$$p_{r:n}(x) = \sum_{k=0}^{r-1} \sum_{m=0}^{n-r} \frac{1}{(r-k-1)!\,(k+m+1)!\,(n-r-m)!} \operatorname{per}\left[\underbrace{P(x-1)}_{r-k-1} \quad \underbrace{p(x)}_{k+m+1} \quad \underbrace{1-P(x)}_{n-r-m}\right] \quad (43)$$

Where

$$P(x-1)=(P_1(x-1),\ P_2(x-1),\dots,P_n(x-1))';$$
$$p(x-1)=(p_1(x),\ p_2(x),\dots,p_n(x))'\ and$$
$$1-P(x)=(1-P_1(x),\ 1-P_2(x),\dots,1-P_n(x))' \quad (44)$$

are column vectors and $P_i(x-1)=0$ when $x_i=0$;
When n is large and considering the case $k=m=0$ i.e. the chance of ties is zero and all the observations are distinct an easier form of the pmf of $X_{r:n}$ is now obtained:

$$P_{r:n}(x)=\frac{\sum_P}{(r-1)!\,(n-r)!}\prod_{a=1}^{r-1}P_{i_a}(x-1)\ p_{i_r}(x)\prod_{c=r+1}^{n}\left(1-p_{i_c}(x)\right) \quad (45)$$

Using the permanent notation it can be written as:

$$P_{r:n}(x)=\frac{\sum_P}{(r-1)!(n-r)!}\ \text{per}\left[\underbrace{P(x-1)}_{r-1}\quad\underbrace{p(x)}_{1}\quad\underbrace{1-P(x)}_{n-r}\right] \quad (46)$$

The pmf of the smallest and largest os $X_{1:n}, X_{n,n}$, considering the cases when $k=m=0$, are respectively in the form:

$$p_{1:n}(x)=\frac{\sum_P}{(n-1)!}\ p_{i_1}(x)\ \prod_{c=2}^{n}\left(1-P_{i_1}(x)\right) \quad (47)$$

and with permanents notation it is written as:

$$p_{1:n}(x)=\frac{1}{(n-1)!}\ \text{per}\left[\underbrace{p\,(x)}_{1}\quad\underbrace{1-P(x)}_{n-1}\right] \quad (48)$$

$$p_{n:n}(x)=\frac{\sum_P}{(n-1)!}\ \prod_{a=1}^{n-1}P_{i_a}(x-1)p_{i_n}(x) \quad (49)$$

and with permanents notation it is written as:

$$p_{n:n}(x)=\frac{1}{(n-1)!}\ \text{per}\left[\underbrace{P(x-1)}_{n-1}\quad\underbrace{p(x)}_{1}\right] \quad (50)$$

The cdf of the r^{th} os , $X_{r:n}$, for inid discrete rv's is given by:

$$P_{r:n}(x)=\text{pr}\left(X_{r:n}\le x\right)$$
$$=\sum_{x=0}^{x}\sum_{k=0}^{r-1}\sum_{m=0}^{n-r}\frac{\sum_P}{(r-k-1)!\,(k+m+1)!\,(n-r-m)!}\prod_{a=1}^{r-k-1}P_{i_a}(x-1)\prod_{b=r-k}^{k+m+1}p_{i_b}(x)\prod_{c=k+m+2}^{n}\left(1-P_{i_c}(x)\right) \quad (51)$$

and when using permanents it is in the form

$$P_{r:n}(x)=\sum_{x=0}^{x}\sum_{k=0}^{r-1}\sum_{m=0}^{n-r}\frac{1}{(r-k-1)!\,(k+m+1)!\,(n-r-m)!}\ \text{per}\left[\underbrace{P(x-1)}_{r-k-1}\quad\underbrace{p(x)}_{k+m+1}\quad\underbrace{1-P(x)}_{n-r-m}\right] \quad (52)$$

and when $k=m=0$ we get :

$$P_{r:n}(x)=\sum_{x=0}^{x}\frac{\sum_P}{(r-1)!\,(n-r)!}\prod_{a=1}^{r-1}P_{i_a}(x-1)p_{i_r}(x)\prod_{c=r+1}^{n}\left(1-P_{i_c}(x)\right) \quad (53)$$

and with permanents notation it is written as:

$$P_{r:n}(x)=\sum_{x=0}^{x}\frac{1}{(r-1)!\,(n-r)!}\text{per}\left[\underbrace{P(x-1)}_{r-1}\quad\underbrace{p(x)}_{1}\quad\underbrace{1-P(x)}_{n-r}\right] \quad (54)$$

Another form of the cdf of $X_{r:n}$ given by (Arnold, B.C. , Balakrishnan, N. and Nagarag, H.N. 1992) for the iid case can be generalized to inid case. The form is given by*:*

$$P_r(x) = \sum_{j=r}^{n} \sum_{p_j} \prod_{a=1}^{j} P_{i_a}(x) \prod_{b=j+1}^{n} [1 - P_{i_b}(x)] \quad (55)$$

When $\sum_P$ denotes the sum over all n! permutations $(i_1, i_2, \dots, i_j)$ of (1,2,… ,n) , such that $i_1 < i_2 < \dots < i_j < i_{j+1} < i_{j+2} < \cdots < i_i$, and using the permanent form it is written as:

$$P_r(x) = \sum_{j=r}^{n} \frac{1}{j!(n-j)!} \text{per} \begin{bmatrix} P_1(x) & 1 - P_1(x) \\ \vdots & \vdots \\ \underbrace{P_n(x)}_{j} & \underbrace{1 - P_n(x)}_{n-j} \end{bmatrix} \quad (56)$$

The above two forms of the cdf will be used for the distribution studied in this paper in calculations of moments. The cdf of the smallest $X_{1:n}$ and largest $X_{n:n}$ order statistics are respectively

$$P_{1:n}(x) = \sum_{t=0}^{x} \frac{1}{(n-1)!} p_{i_1}(t) \prod_{c=2}^{n} \left(1 - P_{i_c}(t)\right) \quad (57)$$

$$P_{1:n}(x) = \sum_{t=0}^{x} \frac{1}{(n-1)!} \text{per} \left[\underbrace{p_1(t)}_{1} \quad \underbrace{1 - P(t)}_{n-1} \right] \quad (58)$$

$$P_{1:n}(x) = \sum_{j=1}^{n} \sum_{p_j} \prod_{a=1}^{j} p_{i_a}(x) \prod_{a=j+1}^{n} [1 - P_{i_a}(x)]$$

$$= 1 - \prod_{a=1}^{n} \left(1 - P_{i_a}(x)\right) \quad (59)$$

$$P_{1:n}(x) = \sum_{j=1}^{n} \frac{1}{j!(n-j)!} \text{per} \left[\underbrace{P(x)}_{j} \quad \underbrace{1 - P(x)}_{n-j} \right] \quad (60)$$

$$P_{n:n}(x) = \sum_{t=0}^{x} \frac{\sum_P}{(n-1)!} \prod_{a=1}^{n-1} P_{i_a}(t-1) p_{i_n}(t)$$

$$P_{n:n}(x) = \sum_{t=0}^{x} \frac{1}{(n-1)!} \text{per} \left[\underbrace{P(t-1)}_{n-1} \quad \underbrace{p(t)}_{1} \right] \quad (61)$$

$$P_{n:n}(x) = \prod_{a=1}^{n} P_{i_a}(x)$$

$$= \frac{1}{n!} \text{per} \left[\underbrace{P(x)}_{n} \right] \quad (62)$$

4.1.1 Distribution of Single OS From INID DGD (2,q) RV's

The pmf of inid $X_{r:n}$ having DGD (2,q) is obtained by substituting Eq.8 and Eq.9 in eq.42 to get :

$$p_{r:n}(x) = \sum_{k=0}^{r-1} \sum_{m=0}^{n-r} \frac{\sum_P}{(r-k-1)!(k+m+1)!(n-r-m)!}$$

$$\prod_{a=1}^{r-k-1} \left(1 - q_{i_a}^{x-1} - (x-1)q_{i_a}^{x-1} + (x-1)q_{i_a}^{x}\right)$$

$$\prod_{b=r-k}^{k+m+1} \left(x\, p_{i_b}^2 q^{x-1}\right) \prod_{c=k+m+2}^{n} \left(q_{i_c}^x + x q_{i_c}^x - x q_{i_c}^{x+1}\right) \quad (63)$$

Using permanent notation it can be written as:

$$p_{r:n}(x) = \sum_{k=0}^{r-1} \sum_{m=0}^{n-r} \frac{\sum_P}{(r-k-1)!(k+m+1)!(n-r-m)!}$$

$$\text{per} \left[\underbrace{1 - q^{x-1} - (x-1)q^{x-1} + (x-1)q^x}_{r-k-1} \quad \underbrace{x\, p^2 q^{x-1}}_{k+m+1} \quad \underbrace{q^x + xq^x + xq^{x+1}}_{n-r-m} \right] \quad (64)$$

When $k = m = 0$, it is

$$P_{r:n}(x) = \frac{\sum_P}{(r-1)!(n-r)!} \prod_{a=1}^{r-1} \left(1 - q_{i_a}^{x-1} - (x-1)q_{i_a}^{x-1} + (x-1)q_{i_a}^{x}\right)$$

$$x\, p_{i_r}^2 q_{i_r}^{x-1} \prod_{c=r+1}^{n} \left(q_{i_c}^x + x q_{i_c}^x - x q_{i_c}^{x+1}\right) \quad (65)$$

$$P_{r:n}(x) = \frac{1}{(r-1)!\,(n-r)!}$$
$$\text{per}\left[\underbrace{1-q^{x-1}-(x-1)q^{x-1}+(x-1)q^{x}}_{r-1}\ \underbrace{x\,p^{2}q^{x-1}}_{1}\ \underbrace{q^{x}+xq^{x}+xq^{x+1}}_{n-r}\right] \tag{66}$$

4.1.2 Distribution of the Smallest and Largest OS Following INID DGD (2 , q) RV's

The pmf of $X_{1:n}$ and $X_{n:n}$ from inid DGD (2 , q) are given by:

$$p_{1:n}(x) = \frac{1}{(n-1)!}\ \text{per}\left[\underbrace{x\,p_i^{2}q_i^{x-1}}_{1}\ \ \underbrace{q_i^{x}+xq_i^{x}+xq_i^{x+1}}_{n-1}\right] \tag{67}$$

$$p_{n:n}(x) = \frac{1}{(n-1)!}$$
$$\text{per}\left[\underbrace{1-q_i^{x-1}-(x-1)q_i^{x-1}+(x-1)q_i^{x}}_{n-1}\ \ \underbrace{x\,p_i^{2}q_i^{x-1}}_{1}\right] \tag{68}$$

4.1.3 Example 1

For n = 3 the pmf of $X_{1:3}$ *and* f $X_{3:3}$ from inid DGD (2 , q) are given by:

$$p_{1:3}(x) = \frac{1}{2!}\ \text{per}\left[\underbrace{x\,p_i^{2}q_i^{x-1}}_{1}\ \ \underbrace{q_i^{x}+xq_i^{x}+xq_i^{x+1}}_{2}\right]$$

$$= \frac{1}{2}\ \text{per}\begin{bmatrix} x\,p_1^{2}q_1^{x-1} & q_1^{x}+xq_1^{x}-xq_1^{x+1} & q_1^{x}+xq_1^{x}-xq_1^{x+1} \\ x\,p_2^{2}q_2^{x-1} & q_2^{x}+xq_2^{x}-xq_2^{x+1} & q_2^{x}+xq_2^{x}-xq_2^{x+1} \\ x\,p_3^{2}q_3^{x-1} & q_3^{x}+xq_3^{x}-xq_3^{x+1} & q_3^{x}+xq_3^{x}-xq_3^{x+1} \end{bmatrix}$$

$$= x(q_1q_2q_3)^{x}\left[\frac{p_1^{2}(xq_2-x-1)(xq_3-x-1)}{q_1} + \frac{(xq_1-x-1)[p_2^{2}q_3(xq_3-x-1)+p_3^{2}q_2(xq_2-x-1)]}{q_2q_3}\right] \tag{69}$$

$$p_{3:3}(x) = \frac{1}{2}\ \text{per}\left[\underbrace{1-q_i^{x-1}-(x-1)q_i^{x-1}+(x-1)q_i^{x}}_{2}\ \ \underbrace{x\,p_i^{2}q_i^{x-1}}_{1}\right]$$

$$= \frac{1}{2}\ \text{per}\begin{bmatrix} 1-q_1^{x-1}-(x-1)q_1^{x-1}+(x-1)q_1^{x} & 1-q_1^{x-1}-(x-1)q_1^{x-1}+(x-1)q_1^{x} & x\,p_1^{2}q_1^{x-1} \\ 1-q_2^{x-1}-(x-1)q_2^{x-1}+(x-1)q_2^{x} & 1-q_2^{x-1}-(x-1)q_2^{x-1}+(x-1)q_2^{x} & x\,p_2^{2}q_2^{x-1} \\ 1-q_3^{x-1}-(x-1)q_3^{x-1}+(x-1)q_3^{x} & 1-q_3^{x-1}-(x-1)q_3^{x-1}+(x-1)q_3^{x} & x\,p_3^{2}q_3^{x-1} \end{bmatrix}$$

$$= \frac{x}{q_1q_2q_3}\{p_1^{2}q_1^{x}(q_2+q_2^{x}((x-1)q_2-x))(q_3+q_3^{x}((x-1)q_3-x)$$
$$+\left[q_1+q_1^{x}((x-1)q_1-x)\right][(\,p_3^{2}\ \left(q_2+q_2^{x}((x-1)q_2-x)\right)q_3^{x}+p_2^{2}\,q_2^{x}(q_3$$
$$+q_3^{x}((x-1)q_3-x))\Big]\} \tag{70}$$

The cdf of $X_{r:n}$ from inid DGD (2, q) is given by substituting (9) in (55) to get:

$$P_{r:n}(x) = \sum_{j=r}^{n}\sum_{p_j}\prod_{a=1}^{j}[1-\ q_{i_a}^{x}(1+x)+xq_{i_a}^{x+1}]$$
$$\prod_{b=j+1}^{n}\left[q_{i_b}^{x}\,(1+x)-x\,q_{i_b}^{x+1}\right] \tag{71}$$

$$P_{r:n}(x) = \sum_{j=r}^{n}\frac{1}{j!(n-j)!}\ \text{per}\left[\underbrace{1-\ q_{i_a}^{x}(1+x)+xq_{i_a}^{x+1}}_{j}\ \ \underbrace{q^{x}(1+x)-xq^{x+1}}_{n-j}\right] \tag{72}$$

Then the cdf of $X_{1:n}$, $X_{n:n}$ are:

$$P_{1:n}(x) = \sum_{j=1}^{n}\sum_{p}\prod_{a=1}^{j}\left(1-\ q_{i_a}^{x}(1+x)+xq_{i_a}^{x+1}\right)$$
$$\prod_{b=j+1}^{n}\left(q_{i_p}^{x}-xq_{i_p}^{x}+xq_{i_b}^{x+1}\right) \tag{73}$$

$$P_{1:n}(x) = \sum_{j=1}^{n}\frac{1}{j!(n-j)!}\ \text{per}\left[\underbrace{1-\ q_{i_a}^{x}(1+x)+xq_{i_a}^{x+1}}_{j}\ \ \underbrace{q_{i_j}^{x}-xq_{i_j}^{x}+xq_{i_j}^{x+1}}_{n-j}\right] \tag{74}$$

$$P_{n:n}(x) = \prod_{a=1}^{n}\left(1-q_{a_i}^{x}-xq_{a_i}^{x}+xq_{i_a}^{x+1}\right)$$

$$= \frac{1}{n!}\ \text{per}\left[\underbrace{1 - q_{i_n}^{x}(1+x) + x\,q_{i_n}^{x} + xq_{i_j}^{x+1}}_{n}\right] \quad (75)$$

4.1.4 Example 2

For n = 3 the cdf of $X_{1:3}$ from inid DGD (2, q) is given by:

$$P_{1:3}(x) = \sum_{j=1}^{3} \frac{1}{j!\,(n-j)!}\ \text{per}\left[\underbrace{1 - q_{i_a}^{x}(1+x) + xq_{i_a}^{x+1}}_{j}\quad \underbrace{q_{i_j}^{x} - xq_{i_j}^{x} + xq_{i_j}^{x+1}}_{3-j}\right]$$

$$= \frac{1}{2}\ \text{per}\left[\underbrace{1 - q_{i_a}^{x}(1+x) + xq_{i_a}^{x+1}}_{1}\quad \underbrace{q_{i}^{x} - xq_{i}^{x} + xq_{i}^{x+1}}_{2}\right]$$

$$+ \frac{1}{2}\ \text{per}\left[\underbrace{1 - q_{i_a}^{x}(1+x) + xq_{i_a}^{x+1}}_{2}\quad \underbrace{q_{i}^{x} - xq_{i}^{x} + xq_{i}^{x+1}}_{1}\right]$$

$$+ \frac{1}{3!}\ \text{per}\left[\underbrace{1 - q_{i_a}^{x}(1+x) + xq_{i_a}^{x+1}}_{3}\right]$$

$$= \frac{1}{2}\ \text{per}\begin{bmatrix} 1 - q_1^x(1+x) + xq_1^{x+1} & q_1^x + xq_1^x - xq_1^{x+1} & q_1^x + xq_1^x - xq_1^{x+1} \\ 1 - q_2^x(1+x) + xq_2^{x+1} & q_2^x + xq_2^x - xq_2^{x+1} & q_2^x + xq_2^x - xq_2^{x+1} \\ 1 - q_3^x(1+x) + xq_3^{x+1} & q_3^x + xq_3^x - xq_3^{x+1} & q_3^x + xq_3^x - xq_3^{x+1} \end{bmatrix}$$

$$= \frac{1}{2}\ \text{per}\begin{bmatrix} 1 - q_1^x(1+x) + xq_1^{x+1} & 1 - q_1^x(1+x) + xq_1^{x+1} & q_1^x + xq_1^x - xq_1^{x+1} \\ 1 - q_2^x(1+x) + xq_2^{x+1} & 1 - q_2^x(1+x) + xq_2^{x+1} & q_2^x + xq_2^x - xq_2^{x+1} \\ 1 - q_3^x(1+x) + xq_3^{x+1} & 1 - q_3^x(1+x) + xq_3^{x+1} & q_3^x + xq_3^x - xq_3^{x+1} \end{bmatrix}$$

$$= \frac{1}{6}\ \text{per}\begin{bmatrix} 1 - q_1^x(1+x) + xq_1^{x+1} & 1 - q_1^x(1+x) + xq_1^{x+1} & 1 - q_1^x(1+x) + xq_1^{x+1} \\ 1 - q_2^x(1+x) + xq_2^{x+1} & 1 - q_2^x(1+x) + xq_2^{x+1} & 1 - q_2^x(1+x) + xq_2^{x+1} \\ 1 - q_3^x(1+x) + xq_3^{x+1} & 1 - q_3^x(1+x) + xq_3^{x+1} & 1 - q_3^x(1+x) + xq_3^{x+1} \end{bmatrix}$$

$$= 1 + q_1^x\, q_2^x\, q_3^x\, (xq_1 - x - 1)(xq_2 - x - 1)\,(x\,q_3 - x - 1) \quad (76)$$

For n = 3 the cdf of $X_{3:3}$ from inid DGD (2, q) is given by:

$$P_{3:3}(x) = \frac{1}{3!}\ \text{per}\left[\underbrace{1 - q_{i_a}^{x}(1+x) + xq_{i_a}^{x+1}}_{3}\right]$$

$$= \frac{1}{6}\ \text{per}\begin{bmatrix} 1 - q_1^x(1+x) + xq_1^{x+1} & 1 - q_1^x(1+x) + xq_1^{x+1} & 1 - q_1^x(1+x) + xq_1^{x+1} \\ 1 - q_2^x(1+x) + xq_2^{x+1} & 1 - q_2^x(1+x) + xq_2^{x+1} & 1 - q_2^x(1+x) + xq_2^{x+1} \\ 1 - q_3^x(1+x) + xq_3^{x+1} & 1 - q_3^x(1+x) + xq_3^{x+1} & 1 - q_3^x(1+x) + xq_3^{x+1} \end{bmatrix}$$

$$= (1 + q_1^x(xq_1 - x - 1))(1 + q_2^x(xq_2 - x - 1))(1 + q_3^x(xq_3 - x - 1)) \quad (77)$$

5- Moments of OS From INID Discrete RV's

First, we consider the case when there is a finite number M which is the largest values of the variate, i.e. x = 0,1,2, … , M .

Since P (M) = 1 ,

$$\therefore \mathrm{E}\left[X_{r:n}^{k}\right] = \mu_{r:n}^{(k)}(x) = \sum_{x=0}^{M} X^{k} p_{i:n}(x)$$

$$= \sum_{x=0}^{M} x^{k}\left[P_{r:n}(x) - P_{r:n}(x-1)\right]$$

$$= M^{k} - \sum_{x=0}^{M-1} P_{r:n}(x)\left[(x+1)^{k} - x^{k}\right] \quad (78)$$

The mean of $X_{r:n}$ is given by:

$$\mu_{r:n}(x) = M - \sum_{x=0}^{M-1} P_{r:n}(x)\left[(x+1) - x\right]$$

$$= M - \sum_{x=0}^{M-1} P_{r:n}(x)$$

$$= \sum_{x=0}^{M-1}\left\{1 - P_{r:n}(x)\right\} \quad (79)$$

Where $P_{r:n}(x)$ is defined in eq.51, through eq.56.

The mean of the smallest and largest order statistics in the inid case are given by:

$$\mu_{1:n}(x) = \sum_{x=0}^{M-1} \left\{1 - P_{1:n}(x)\right\} = \sum_{x=0}^{M-1} \left\{ \prod_{i=1}^{n} (1 - P_i(x)) \right\} \qquad (80)$$

$$\mu_{n:n}(x) = \sum_{x=0}^{M-1} \left\{1 - P_{n:n}(x)\right\} = \sum_{x=0}^{M-1} \left\{ 1 - \prod_{a=1}^{n} P_{i_a}(x) \right\} \qquad (81)$$

5.1 Moments of OS From INID DGD(2,q) RV's

The kth moment of $X_{r:n}$ from inid DGD (2, q) is obtained by substituting (55) in (78) to get:

$$\mu_{r:n}^{(k)} = M^k - \sum_{x=0}^{M-1} \sum_{j=r}^{n} \sum_{p_j} \prod_{a=1}^{j} [1 - q_{i_a}^{x}\,(1+x) + x\, q_{i_a}^{x+1}] \prod_{b=j+1}^{n} [q_{i_b}^{x}\,(1+x) - x\, q_{i_b}^{x+1}] * [(x+1)^k - x^k] \qquad (82)$$

The mean of the rth os is:

$$\mu_{r:n}^{(k)} = \sum_{x=0}^{M-1} \sum_{j=r}^{n} \sum_{p_j} \prod_{a=1}^{j} [1 - q_{i_a}^{x}\,(1+x) + x\, q_{i_a}^{x+1}] \prod_{b=j+1}^{n} [q_{i_b}^{x}\,(1+x) - x\, q_{i_b}^{x+1}] \qquad (83)$$

And the mean for the smallest and largest os from inid DGD (2, q) are given by:

$$\mu_{1:n} = \sum_{x=0}^{M-1} \prod_{i=1}^{n} [q_i^x (1+x) - x\, q_i^{x+1}] \qquad (84)$$

$$\mu_{n:n} = \sum_{x=0}^{M-1} \left(1 - \prod_{i=1}^{n} [1 - q_i^x (1+x) + x\, q_i^{x+1}] \right) \qquad (85)$$

5.1.1 Example3

For M=3, n=3, the mean of the smallest and largest os are then:

$$\mu_{1:3} = \sum_{x=0}^{2} \prod_{i=1}^{3} [q_i^x (1+x) - x\, q_i^{x+1}] \qquad (86)$$

$$\mu_{3:3} = \sum_{x=0}^{2} \left(1 - \prod_{i=1}^{3} [1 - q_i^x (1+x) + x\, q_i^{x+1}] \right) \qquad (87)$$

Calculations of the mean of the smallest and largest os from DGD (2,q) are given in Table 1. and Table 2. for several values of q_1, q_2, q_3:

Table 1. The Mean of the Smallet OS From INID DGD (2,q).

$q_3 = 0.1\,(\,0.1\,)\;0.5$					
q_1 \ q_2	0.1	0.2	0.3	0.4	0.5
0.1	1.00688 1.01308 1.01858 1.02338 1.02747	1.01308 1.02493 1.03551 1.0448 1.05276	1.01858 1.03551 1.05073 1.06414 1.0757	1.02338 1.0448 1.06414 1.08129 1.09613	1.02747 1.05276 1.0757 1.09613 1.11388
0.2	1.01308 1.02493 1.03551 1.0448 1.05276	1.02493 1.04778 1.06843 1.08675 1.10261	1.03551 1.06843 1.09849 1.12541 1.14893	1.0448 1.08675 1.12541 1.16034 1.1911	1.05276 1.10261 1.14893 1.1911 1.2285

0.3	1.01858	1.03551	1.05073	1.06414	1.0757
	1.03551	1.06843	1.09849	1.12541	1.14893
	1.05073	1.09849	1.14273	1.18289	1.2184
	1.06414	1.12541	1.18289	1.23566	1.28282
	1.0757	1.14893	1.2184	1.28282	1.34088
0.4	1.02338	1.0448	1.06414	1.08129	1.09613
	1.0448	1.08675	1.12541	1.16034	1.1911
	1.06414	1.12541	1.18289	1.23566	1.28282
	1.08129	1.16034	1.23566	1.30576	1.36915
	1.09613	1.1911	1.28282	1.36915	1.448
0.5	1.02747	1.05276	1.0757	1.09613	1.11388
	1.05276	1.10261	1.14893	1.1911	1.2285
	1.0757	1.14893	1.2184	1.28282	1.34088
	1.09613	1.1911	1.28282	1.36915	1.448
	1.11388	1.2285	1.34088	1.448	1.54688

Table 2. The Mean of the Largest OS From INID DGD (2,q).

$q_3 = 0.1\,(\,0.1\,)\;0.5$					
q_1 \ q_2	0.1	0.2	0.3	0.4	0.5
0.1	1.55023	1.73357	1.9378	2.15158	2.36358
	1.73357	1.88789	2.06319	2.24903	2.43494
	1.9378	2.06319	2.20807	2.36331	2.51975
	2.15158	2.24903	2.36331	2.48688	2.61217
	2.36358	2.43494	2.51975	2.61217	2.70638
0.2	1.73357	1.88789	2.06319	2.24903	2.43494
	1.88789	2.01853	2.16989	2.33232	2.49619
	2.06319	2.16989	2.2956	2.43191	2.57037
	2.24903	2.33232	2.43191	2.54082	2.6521
	2.43494	2.49619	2.57037	2.6521	2.736
0.3	1.9378	2.06319	2.20807	2.36331	2.51975
	2.06319	2.16989	2.2956	2.43191	2.57037
	2.20807	2.2956	2.40046	2.51527	2.63265
	2.36331	2.43191	2.51527	2.60729	2.70188
	2.51975	2.57037	2.63265	2.70188	2.77337
0.4	2.15158	2.24903	2.36331	2.48688	2.61217
	2.24903	2.33232	2.43191	2.54082	2.6521
	2.36331	2.43191	2.51527	2.60729	2.70188
	2.48688	2.54082	2.60729	2.68125	2.75765
	2.61217	2.6521	2.70188	2.75765	2.8155
0.5	2.36358	2.43494	2.51975	2.61217	2.70638
	2.43494	2.49619	2.57037	2.6521	2.736
	2.51975	2.57037	2.63265	2.70188	2.77337
	2.61217	2.6521	2.70188	2.75765	2.8155
	2.70638	2.736	2.77337	2.8155	2.8593

6- Conclusion

Order statistics from discrete version of the continuous gamma distribution is proposed for the iid and inid cases. Forms and graphs of its pmf and cdf are studied. It is regarded that like the continuous gamma distribution, the present discrete gamma distribution is useful in discrete failure data analysis and also in modeling discrete data from other fields as well. Concentration should be paid for the subject of order statistics (os) for other new discretized distributions and also for discontinuous distributions.

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