Characterization of Banach spaces to have the approximation property

Osama Tantawy, Alaa Eldeen Hamza Sayed and Mahmoud Zaghrout

Mathematics Dept., Zagazig University, Zagazig, Egypt
Zmm.mahmoud@yahoo.com

Abstract: In this paper we study some results about the approximation property on Banach spaces (B-Spaces). We give a sufficient condition using trace mapping for a Banach space to posses metric approximation property. We also study the approximation property for dual spaces. We prove that a Banach sapce has the bounded approximation property if the identity operator belongs to the closure of the collection of bounded and finite rank operators with the weak operator topology. Also a Banach space has the bounded weak approximation property if every compact operator on it belongs to the closure of the collection of finite rank operators with the weak operator topology.

[Osama Tantawy, Alaa Eldeen Hamza Sayed and Mahmoud Zaghrout. **Characterization of Banach spaces to have the approximation property** *J Am Sci* 2013;9(11):480-484]. (ISSN: 1545-1003). http://www.jofamericanscience.org. 61

Keywords: approximation property**,** dual spaces

1.Introduction:

The study of the approximation property is important in Banach space theory. The approximation property due to the following fundamental result in the theory of operators on Hilbert spaces: if X and Y are Hilbert spaces, then the compact operators from Y to X are the norm closure of the finite – rank maps. The initial conjecture was that every Banach space satisfies the approximation property.

Let X and Y be Banach spaces. In this paper we use the following notations:

L(X,Y) the space of bounded linear operators from X to Y.

F(X, Y) the space of bounded and finite rank operators from X to Y.

C(X, Y) the space of compact operators from X to Y.

C^*(X, Y)= $\{T^*: T: X \to Y\}$.

C(X, Y, λ) the space of compact operators from X to Y satisfying $\|T\| \leq \lambda$ where 1≤ λ< ∞.

In the following, we will denote for isomorphic between B-spaces by $\cong$

1. We say that a Banach space has the approximation property (in short AP), if every compact operator is a limit of finite rank operators. In fact, this notion can be formulate in an equivalent definition as follows: A Banach space Y possess the approximation property if for every $\epsilon > 0$ and every compact set K in Y there is a finite rank operator $T: Y \to Y$ such that $\|Ty - y\| \leq \epsilon$ for every y $\in K$, i.e, a bounded linear operator of finite rank operators can be represented as $Ty = \sum_{i=1}^{n} y_i^*$ (y) y_i , for some $\{y_i^*\}_{i=1}^{n}$ in Y^* and $\{y_i\}_{i=1}^{n}$ in Y.

Notice that: A closed subset K of a Banach space Y is compact if and only if there is a sequence $\{y_n\}_{n=1}^{\infty}$ in Y such that $\|y_n\| \to 0$ and $K \subset \overline{conv}\{y_n\}_{n=1}^{\infty}$.See[1]

Before we formulate the theorems of the approximation property, we need the following theorem.

Theorem 1.1 [1]

Suppose X and Y are Banach spaces, define on L (X, Y) the topology τ of uniform convergence on compact sets in X. Then, the continuous linear functionals on (L(X, Y), τ) consist of all functionals φ of the form $\varphi(T) = \sum_{i=1}^{\infty} y_i^*(Tx_i)$, for some $\{y_i^*\}_{i=1}^{\infty} \subset Y^*$,$\{x_i\}_{i=1}^{\infty} \subset X$ with $\sum_{i=1}^{\infty} \|y_i^*\| \|x_i\| < \infty$.

The topology τ on L (X,Y) of compact convergence is the locally convex topology generated by the seminorms of the form $\| T \|_k$=sup $\{\|Tx\|$: x $\in K\}$, where K ranges over all compact subsets of X $\forall T \in L(X, Y)$. Besides, the dual space $(L(X, Y), \tau)^*$ can be identified exactly with the projective tensor product $Y^* \hat{\otimes} X$.

We remark that a B-space X has the approximation property if $I_X \in \overline{F(X, X)}^{\tau}$, where I_X is identity operator on X [2].

For Y^* we have the following:

Theorem 1.2

There is a Banach space Y with a boundedly complete basis such that its dual Y^* of Y is separable and does not have the AP.

To prove theorem 1.2 we need the following result of Lindenstrauss.

Lemma 1.1.

If V is a separable Banach space, then there is a separable Banach space W such that W^{**} has a boundedly complete basis, $W^{**}/W \cong V$, and $W^{***} \cong W^* \oplus V^*$.

Proof of the theorem 1.2

The proof of the theorem 1.2 depends on the fact that there is a Banach space which does not have the AP. let Y be separable Banach space which does not have the AP. By lemma 1.1 there is a separable Banach space X such that its X^{**} has a basis such that $X^{**}/X \cong Y$ and $X^{***} \cong X^* \bigoplus Y^*$, where there is a projection from X^{***} into X^*. This projection is the map which determines to every functional on X^{**} its restriction to X. Since Y does not have the AP, then the Y^* does not have the AP. Any complemented subspace of a space having the AP must have the AP, therefore X^{***} which is a dual of a space X^{**} with a basis, does not have the AP. Since Y and Y^* are separable, then X^{***} is separable. □

here we clarify the definition of X, let $\{y_n\}_{n=1}^{\infty}$ be a sequence which is dense on the boundary of the unit ball of Y. The space X consists of all the sequences $x = (a_1, a_2, \ldots..)$ of scalars for which

(1) $\|x\| = \sup \left(\sum_{j=1}^{m} \| \sum_{j=p_{j-1}+1}^{p_j} a_i y_i \|^2\right)^{\frac{1}{2}} < \infty$

(2) $\sum_{i=1}^{\infty} a_i y_i = 0$

The supremum in (1) is taken over all choices of integers m and

$$p_0 < p_1 < \ldots. < p_m.$$

Another construction of space X having the properties required in theorem 1.2 is given in [3]. [3] contains the construction of space X satisfying $X^{**} / X \cong Y$ also for a large natural class of non–separable space Y. Every space X obtained in theorem 1.2 is not reflexive but has a separable second dual.

2- We shall investigate the approximation property by constant independent of compact set K.

Let λ≥ 1, we say that a Banach space Y has the λ-approximation property (λ–AP in short), if for every compact set K⊂ Y and every ϵ>0, there is an operator $T: Y \to Y$of finite rank such that $\|Ty - y\| \le \epsilon$ for every y∈ K and $\|T\| \le \lambda$. We say that a Banach space Y has bounded approximation property (BAP in short) if it has the λ–AP for some λ and has metric approximation property (MAP in short) if it has 1-AP.

The next result clarifies the relation between the AP and the MAP.

Theorem 2.1

Let Y be a separable space which is isometric to a dual space and which has the AP then Y has the MAP.

For the proof of 2.1 see [4].

It follows from 2.1 that, for separable reflexive space, the AP implies the MAP. The same is true for non separable reflexive space.

In the fact, for general the BAP does not imply the MAP and the AP does not imply the BAP this was shown by [5].

We say that X has the weak approximation property (in short, WAP) if for every $T \in C(X)$, compact K⊂X and $\epsilon > 0$, there is a $T_0 \in F(X)$ such that $\|T_0 x - Tx\| \le \epsilon$ for all x ∈ K. We say that X has the quasi-approximation property (in short, QAP) if for every $T \in C(X)$, and $\epsilon > 0$, there is a $T_0 \in F(X)$ such that $\|T_0 - T\| \le \epsilon$. Now we consider dual problems for approximation properties. It is well known that the AP, BAP, and MAP are not inherited from X to X^*. See [6], also in [6] it was shown that the WAP, BWAP and QAP are not inherited from X toX^*.

Grothendieck [4] systematically investigated the AP and showed the following fact:

(a) X has the AP if and only if for every Banach space Y, C(Y, X) = $\overline{F(Y, X)}$.

(b) X^* has the AP if and only if for every Banach space Y, C(X, Y) = $\overline{F(X, Y)}$.

Now we introduce a characterization of the AP.

Lemma 2.1

Y has the approximation property iff for every Banach space X, C(X,Y,1) = F(X,Y,1).

Proof

Suppose that Y has the AP. Let X be a B-space with $T \in C(X, Y, 1)$ and ϵ > 0. Let $\lambda > 0$ with $\frac{\lambda}{1+\lambda} < \frac{\epsilon}{2}$.

Since Y has the AP, by fact (a) there is $T_0 \in F(X, Y)$ such that

$$\|T_0 - T\| < \lambda.$$

Then we see $T_0 \in F(X, Y, 1 + \lambda)$, and define $f_0 = \frac{1}{(1+\lambda)T_0}$. Then $f_0 \in F(X, Y, 1)$ with $\|f_0 - T\| \le \frac{1}{1+\lambda}\|T_0 - T\| + \frac{\lambda}{1+\lambda}\|T\| \le$ ϵ. Hence $T \in \overline{F(X, Y, 1)}$.

Conversely: f_0 use fact (a), let X be B-space and $T \in C(X, Y)$, then by assumption we have

$T \in C(X, Y, \|T\|) = \|T\| C(X, Y, 1) = \|T\| \overline{F(X, Y, 1)} = \overline{F(X, Y, \|T\|)} \subset \overline{F(X, Y)}$.

Hence $T \in \overline{F(X, Y)}$ and the proof is complete. □

- Let S be the trace mapping from the projective tensor product $Y^* \widehat{\otimes} Y$ *to* F(Y, Y)*, the dual space of F(Y, Y) is defined by

(S_u)(T) = trace (T_u), $u \in Y^* \widehat{\otimes} Y, T \in F(Y, Y)$, that is, if $u = \sum_{n=1}^{\infty} y_n^* \otimes y_n$ then $(S_u)(T) = \sum_{n=1}^{\infty} y_n^*(Ty_n)$. We shall always regard Y as a subspace of Y^{**}. Thus the identity operator I_Y on Y is also considered as embedding identifying I_Y with canonical embedding $J_Y: Y \to Y^{**}$. The following results hold for the general version of the metric approximation property defined by operator ideal $\mathcal{A}$ (in the sense of pietsch, see [7]). In Banach space Y we denote the closed unit ball by B_Y. We say that a Banach space Y has the metric $\mathcal{A}$ approximation property (M $\mathcal{A}$ A P), if for every compact set K in Y and every $\epsilon > 0$ there is an operator T$\in B_{\mathcal{A}(Y,Y)}$ such that $\|Ty - y\| \le \epsilon$ for all y$\in K$.

Notice that:

$\mathcal{A}(Y,Y)$ is equipped with the norm topology from L(Y,Y). Thus the trace mapping S: $Y^* \widehat{\otimes} Y \to \mathcal{A}(Y,Y)^*$ has norm 1.

Recall that: if Y is a Banach space. The trace mapping S: $Y^* \widehat{\otimes} Y \to \mathcal{A}(Y,Y)^*$ with the condition $I_Y \in S^*(B_{\mathcal{A}(Y,Y)^{**}})$ implies that Y has the M$\mathcal{A}\,A\,P$, where $\mathcal{A}$ is operator ideal. Indeed, the condition $I_Y \in S^*(B_{\mathcal{A}(Y,Y)^{**}})$ clarify by using canonical identification $(Y^* \widehat{\otimes} Y)^*$=L(Y, Y^{**}).

With using the canonical identification $(Y^* \widehat{\otimes} Y)^*$ =L(Y^*, Y^*) the condition becomes equivalent to $I_{Y^*} \in S^*(B_{\mathcal{A}(Y,Y)^{**}})$. Hence, since L(Y, Y^{**}) is canonically identified with L(Y^*, Y^*) under the mapping $T \to T^* \circ j_{Y^*}$ the identity operator I_Y or $j_Y \circ I_Y$ identifies with $(j_Y \circ I_Y)^* \circ j_{Y^*} = I_Y^* \circ j_Y^* \circ j_{Y^*} = I^*{}_Y \circ I_{Y^*} = I_{Y^*}$. Besides, an operator ideal $\mathcal{A}$ is symmetric if $T^* \in \mathcal{A}(Y^*, X^*)$ where T $\in \mathcal{A}(X,Y)$.

Theorem 2.2

With an operator ideal $\mathcal{A}$, a Banach space Y has M$\mathcal{A}\,A\,P$ if the trace mapping S :$Y^* \widehat{\otimes} Y \to \mathcal{A}(Y,Y)^*$ is isometric.

Proof

Since ($Y^* \widehat{\otimes} Y)^* = L(Y, Y^{**})$ we have S*: $\mathcal{A}(Y,Y)^{**} \to L(Y,Y^{**})$ is adjoint of an into isometry, for every $T \in L(Y, Y^{**})$. In particular for T=I_Y there is φ$\in \mathcal{A}(Y,Y)^{**}$ satisfying S$^*\varphi = T$ and $\|\varphi\| = \|T\|$. Hence, $I_Y \in S^*(B_{\mathcal{A}(Y,Y)^{**}})$, this means that Y has the $M\mathcal{A}AP$□

Now, we say that the trace mapping $S : Y \widehat{\otimes} X \to \mathcal{A}(X, Y^*)^*$ is isometric for every Banach space Y, if a Banach space X has the M$\mathcal{A}\,A\,P$, where $\mathcal{A}$. is an operator ideal.

Proposition 2.1

Let Y be a Banach space does not have the $M\mathcal{A}AP$, then the trace mapping $S: Y^* \widehat{\otimes} Y \to \mathcal{A}(Y,Y)^*$ is not isometric, but its dual space Y^* has the $M\mathcal{A}AP$, then $W: Y^* \widehat{\otimes} Y \to \mathcal{A}(Y, Y^{**})^*$ is isometric, where in the two cases $\mathcal{A}$ is a symmetric operator ideal.

Proof

By theorem 2.2, if Y does not have $M\mathcal{A}\,AP$, then S is not isometric. Suppose that Y^* has the $M\mathcal{A}\,AP$. According to above the trace mapping $Y^* \widehat{\otimes} Y \to \mathcal{A}(Y^*, Y^*)$ is isometric. Since $\mathcal{A}$ is a symmetric operator ideal, $\mathcal{A}(Y, Y^{**})$ is canonically identified with $\mathcal{A}(Y^*, Y^*)$ under the mapping $T \to T^* \circ j_{Y^*}$. Hence W is isometric □

3-We say that a Banach space Y has λ-bounded compact approximation property (λ–BCA**P** in short) if for every $\epsilon > 0$ and every compact set $K \subset Y$, there is $T \in C(Y, \lambda)$ such that $\|Ty - y\| \le \epsilon$ for all $y \in K$. If Y has the λ – BCAP for some λ > 0, then Y is said to have the bounded compact approximation property.

A Banach space Y is said to have the bounded weak approximation property (BWAP in short), if for every $T \in C(Y)$, there is $\lambda_T > 0$ such that for every compact set $K \subset Y$ and for every $\epsilon > 0$, there is $T_0 \in F(Y, \lambda_T)$, such that $\|T_0 y - Ty\| < \epsilon$ for all $y \in K$ [8, 9].

For spaces Y and Y^* we have the following results.

Theorm3.1

Let Y be a Banach space and 1 ≤ λ<∞.Then, the following three assertions hold.

(i) Y Possess the λ – BAP iff $I \in \overline{F(Y,\lambda)}^{wo}$

(ii) Y Possess the λ – BCAP iff $I \in \overline{C(Y,\lambda)}^{wo}$

(iii) Y Possess the BWAP iff for each $T \in C(Y)$ there is a λ_T>0 such that $T \in \overline{F(Y,\lambda_T)}^{wo}$.

For Y^* we have following.

Theorem 3.2

Let Y be a Banach space. Then the following three assertions are equivalent:

(i) Y^* Posses the BWAP.

(ii) For every $T \in C(Y^*)$, there exists a $\lambda_T > 0$ so that there exists a net(T_a^*)in $F^*(Y, \lambda_T)$ such that $y^{**}T_a^* y^* \to y^{**}Ty^*$ for every $y^* \in Y^*$ and $y^{**} \in Y^{**}$.

(iii) For every $T \in C(Y^*)$ there is $\lambda_T > 0$ such that there are $(y_i^*)_{i=1}^n \subset Y^*$ and $(y_i^{**})_{i=1}^n \subset Y^{**}$, if $|\sum_{i=1}^n y^{**}(S^* y_i^*)| \le 1$ for all S^* in F^*(Y, λ_T), then $|\sum_{i=1}^n y^{**}(T\, y_i^*)| \le 1$.

To prove the above theorems we need the following topologies and the relation between them.

Definition 3.1

Let X and Y be Banach spaces and let S be the linear span of all linear functionals φ on L(X,Y) of the form φ(T) =y^*Tx for x$\in X$ and $y^* \in Y^*$, then the weak operator topology (wo, in short) on L(X, Y) is topology generated by S.

For a net $(T_a) \subset L$(X, Y) and $T \in L(X,Y), T_a \to T$ in (L(X, Y), wo) *if and only if* for each x$\in X$ and $y^* \in Y^*, y^*T_a x \to y^*Tx$ (1)

Definition 3.2

Let X and Y be Banach spaces, for compact set $K \subset X, \epsilon > 0$ and $T \in L(X,Y)$ we put $\mathcal{B}$ ={R$\in L$(X, Y): $sup_{x \in K}\|Rx - Tx\| < \epsilon$ }.

Let β be the collection of all such $\mathcal{B}(T, K, \epsilon)$. Then, the τ – topology on L(X, Y) is topology generated by β. This topology is called the topology of compact convergence.

By the definitions of BAP, BCAP and BWAP we proved the following:

Y Possess the λ - BAP if and only if $I \in \overline{F(Y,\lambda)}^{\tau}$ (2)

Y Possess the BWAP if and only if $I \in \overline{C(Y,\lambda)}^{\tau}$ (3)

Y Possess the BWAP ⇔for every $T \in C(Y)$ there is $\lambda_T > 0$ such that $T \in \overline{F(Y,\lambda_T)}^{\tau}$. (4)

For a net $(T_a) \subset L(X,Y)$and $T \in L(X,Y), T_a \to T$ in (L(X,Y),τ) *if and only if* for every compapct K⊂ X $sup_{x\in K}\|T_a x - Tx\| \to 0$ (5)

On other hand there is a topology called the topology of pointwise convergence which is defined by the strong operator topology at each of x ∈ X (sto, in short) on L(X, Y).

The next theorem shows the relations between τ, sto and wo.

Theorem 3.3 [8]

Suppose X and Y are Banach spaces, let Z be a bounded subset of L(X, Y), and let C be a convex subset of X, then

(i) τ = sto on Z.

(ii) (L(X,Y), sto)*=(L(X,Y),wo)* and the form of the linear bounded functional φ on L(X,Y) is given by $\varphi(T) = \sum_{i=I}^{n} y_i^*(Tx_i)$, for some $(x_i)_{i=1}^{n} \subset X$ and $(y_i^*)_{i=1}^{n} \subset Y^*$

(iii) X has two locally convex topologies τ_1 and τ_2 such that the dual spaces of X under the two topologies are the same, then the τ_1- closure of C is the same as its τ_2- closure. See [10,11]

Statement 3.1

The following statements are true from theorem 3.5

(1) $\overline{C}^{sto} = \overline{C}^{wo}$ .where C is a convex set in L(X,Y)

(2) $\overline{C}^{\tau} = \overline{C}^{wo}$. where C is a bounded convex set in L(X,Y)

(3) The τ - closure in $F(Y^*)$ can be identified with the τ - closure in $F^*(Y)$ with $\lambda > 0$, that is $\overline{F(Y^*,\lambda)}^{\tau} = \overline{F^*(Y,\lambda)}^{\tau}$.

(4) Since for $\lambda > 0$ $F^*(Y,\lambda)$ is bounded and convex. Then by (2) and (3) we have $\overline{F(Y^*,\lambda)}^{\tau} = \overline{F^*(Y,\lambda)}^{wo}$.

Proof of theorem 3.1

(i) Since F(Y, λ) is bounded and convex for λ>0.there is C in F(Y, λ) is bounded and convex. such that by Statement 3.1(2) we have $\overline{C}^{\tau} = \overline{C}^{wo}$, implies that $\overline{F(Y,\lambda)}^{\tau} = \overline{F(Y,\lambda)}^{wo}$, therefore $I \in \overline{F(Y,\lambda)}^{wo}$. By (2) Y has the λ – BAP iff $I \in \overline{F\ (Y,\lambda)}^{wo}$.

(ii) Since C(Y, λ) is bounded and convex for λ>0 there is subset K in C(Y, λ) is bounded and convex. Then, $\overline{K}^{\tau} = \overline{K}^{wo}$ implies that $\overline{C(Y,\lambda)}^{\tau} = \overline{C(Y,\lambda)}^{wo}$ therefore $\in \overline{C(Y,\lambda)}^{wo}$. By (3) Y has the λ- BCAP iff $I \in \overline{C(Y,\lambda)}^{wo}$.

(iii) From the statement 3.1 (2) we have $\overline{K}^{\tau} = \overline{K}^{wo}$ in C(Y, λ) and $\overline{C}^{\tau} = \overline{C}^{wo}$ in F(Y, λ), hence $\overline{F(Y,\lambda_T)}^{\tau} = \overline{F(Y,\lambda_T)}^{wo}$.

Now T∈ $C(Y)$is the limit of a sequnce of oprators T_i with finite rank. Then, for $\lambda_T > 0$ a $T \in C(Y) \subset \overline{F(Y)}^{\tau}$, that is there is a $\lambda_T > 0$ such that $T \in \overline{F(Y,\lambda_T)}^{\tau} = \overline{F(Y,\lambda_T)}^{wo}$ *implies* $T \in \overline{F(Y,\lambda_T)}^{wo}$, that is Y has the BWAP □

Proof of the theorem 3.2

We notice that Y^* has the BWAP if and only if for every $T \in C(Y^*)$ there is $\lambda_T > 0$ such that $T \in \overline{F^*(Y,\lambda_T)}^{wo}$. By statement 3.6 (3) we have $\overline{F(Y^*,\lambda_T)}^{\tau} = \overline{F^*(Y,\lambda_T)}^{wo}$, and by 3.3 and 3.5, for every $T \in C(Y^*)$ there is a net $(T_\propto) \subset \overline{F^*(Y,\lambda_T)}^{wo} = \overline{F(Y^*,\lambda_T)}^{\tau}$, $T_\alpha^* \to T$ in $\overline{F^*(Y,\lambda_T)}^{wo}$ for $y^* \in Y^*$ and $y^{**} \in Y^{**}$such that $y^{**}T_\alpha^* y^* \longrightarrow y^{**}Ty^*$ so (i)⇔(ii).

In the following we show that (i)⇔(iii). By theorem 3.5 (ii) with $T \in C(Y^*)$ we have $(C(Y^*),sto)^* = (C(Y^*),wo)^*$ and bounded linear functional φ on $C(Y^*)$ is

$\varphi(T) = \sum_{i=1}^{n} y_i^{**}(Ty_i^*)$ for $\{y_i^*\}_{i=1}^{n} \subset Y^*$ and $\{y^{**}\}_{i=1}^{n} \subset Y^{**}$. Since $F^*(Y,\lambda)$ is balanced and convex for $\lambda > 0$, then the set C in $F^*(Y,\lambda)$ is balanced and convex. Hence by the separation theorem (see [11- theorem 2.2.28]) for every $\varphi \in (F^*(Y),wo)^*$ such that $|\varphi(S^*)| \leq 1$ for all S^* in the weak – closure of C in F^* (Y,λ_T), we have $|\varphi(T)| \leq 1$ that is

$$|\varphi(S^*)| = |\sum_{i=1}^{n} y_1^{**}(S^*y^*)| \leq 1 \Longrightarrow |\varphi(T)| = |\sum_{i=1}^{n} y_i^{**}(Ty_i^*)| \leq 1$$ □

References

1. Lindenstrauss J., Lior Tzafrir, Classical Banach spaces I and II, Springer (1977).
2. Kim J.M., New criterion of the approximation property, Journal of mathematical Analysis and Applications, 889-891 (2008).
3. Grothendieck A., Produits tasoriels topoloygiues et space nucleaires, Memo. Amer. math Soc. 16(1955).

4. Lindenstrauss J.. on Jame, paper, Separable conjugate space" Israel J. math 9,279-584, (1971).
5. Davis W.J., T. Figiel, W.B. Johnson, A pelczynski, factoring weakly compact operators, J.funct.Anal 17,311-327(1964).
6. Figiel, T., Johnson, W.B: The approximation property does not imply the bounded approximation property. Proc. Amr. Math. Soc.41, 197-200(1973).
7. Kim J.M., Dual problems for weak and quasi approximation properties, Journal of Mathematical Analysis and Applications, Volume 321, Issue 2, 15 September 569-575 (2006).
8. PIETSCH A.. Operator Ideals. North-Holland Publishing Company, Amsterdam-New York-Oxford (1980).
9. Choi C. and J.M. Kim, Weak and quasi approximation properties in Banach spaces. J. Math. Anal. Appl. 722-735 (2006).
10. Dunford N. and J.T.Schwartz, Linear operators, NewYork, Interscience(1958).
11. Megginson R.E., An Introduction to Banach space Theory, Springer, New York (1998).

11/5/2013