Two-Sample Prediction Of GOS's From Finite Mixture Distributions Based On Generalized Type-II HCS

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Abstract: This paper is concerned with the problem of predicting the future generalized order statistics ($gos's$) based on a mixture of two components from a class of continuous distributions. Generalized Type-II hybrid censoring scheme (HCS) of the observed data has been used here. The prior belief of the experimenter is measured by a general class of distributions, suggested by AL-Hussaini (1999b). We consider the two sample prediction technique to compute Bayesian predictive intervals for a future $gos's$. A mixture of two Weibull components model is considered as a special case of the class. Our results are specialized to upper order statistics and upper record values. Also, we give an example based on real data. Finally, Markov Chain Monte Carlo algorithm is used to find the Bayesian predictive intervals.

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1. Introduction

In many practical problems of statistics, one wishes to use the results of a previous data to predict the results of a future data from the same population. One way to do this is to construct an interval, which will contain these results with a specified probability. This interval is called the prediction interval. Prediction has been applied in medicine, engineering, business and other areas as well. For details on the history of statistical prediction, analysis and applications, see for example, Aitchison and Dunsmore (1975), Geisser (1993), Dunsmore (1974), Howlader and Hossain (1995), AL-Hussaini (1999a, 1999b), Corcuera and Giummol (1999), Nordman and Meeker (2002), Ahmadi *et al.*(2010), Ahmadi *et al.* (2005), Ahmadi and Balakrishnan (2012), Ateya (2011), Ahmad *et al.*(2012), Balakrishnan and Shafay (2012) and Shafay and Balakrishnan (2012).

Several authors have predicted future order statistics and records from homogeneous and heterogeneous populations that can be represented by single-component distribution or finite mixtures of distributions, respectively. For more details, see AL-Hussaini *et al.*(2001), AL-Hussaini and Ahmad (2003a, 2003b), Ali Mousa and AL-Sagheer (2006) and AL-Hussaini (2004).

The two most popular censoring schemes are Type-I and Type-II censoring schemes. The hybrid censoring scheme is the mixture of Type-I and Type-II censoring schemes. It was introduced by Epstein (1954). In hybrid censoring scheme the life-testing experiment is terminated at a random time $T_1^* = \min\{X_{r:n}, T\}$, where $r \in 1,2,...,n$ and $T \in (0,\infty)$ are fixed in advance. Following Childs et al. (2003), we will refer to this scheme as Type-I hybrid censoring scheme (Type-I HCS), since under this scheme the time on test will be no more than T. Recently, it becomes quite popular in the reliability and life-testing experiments, see for example, the work of Chen and Bhattacharya (1988), Gupta and Kundu (1988), Kundu (2007) and Kundu and Howlader (2010).

Noting that this scheme, which would guarantee the experiment to terminate by a fixed time T, may result in few failures, for this reason, Childs et al.(2003) proposed a new HCS, referred to as Type-II hybrid censoring scheme (Type-II HCS), which guarantees a fixed number of failures. Inference based on Type-II HCS from Weibull distribution was made by Banerjee and Kundu (2008). Though the Type-II HCS guarantees a specified number of failures, it has the disadvantage that it might take a very long time to observe r failures and complete the life test.

Chandrasekar, *et al.*(2004) found that both Type-I and Type-II HCS's have some potential drawbacks. Specifically, in Type-I HCS, there may be very few or even no failures observed whereas in Type-II HCS the experiment could last for a very long period of time. So, they suggest generalized hybrid censoring schemes.

Finite mixture of distributions have proved to be of considerable interest in recent years in terms of both the methodological development and multifarious applications. Mixture distribution modeling was studied as early as the early 1890s by Pearson (1894), see also Richardson and Green (1997). Kim and Bai (2002) inferred two component Weibull mixtures from accelerated life test data using maximum likelihood estimates (MLE) with the EM algorithm. Jiang and Murthy (1995, 1998) developed a graphical method for inferring a mixture of two Weibull distributions from failure data. Ahmad *et al.*(1997) found approximate Bayes estimation for mixtures of two Weibull distributions under Type-II censoring. The class of all finite mixtures of Weibull distributions is identifiable, this is proved by Ahmad (1988).

The concept of $gos's$ was introduced by Kamps (1995) as a unified approach to several models of ordered random variables such as upper order statistics ($uos's$), upper record values ($urv's$), sequential order statistics, ordering via truncated distributions and censoring schemes, see for example, Kamps and Gather (1999), AL-Hussaini (2004), Jaheen (2002) and Ahmad (2011).

Let us consider a general class of continuous distributions that suggested by AL-Hussaini and Ahmad (2003a, 2003b) with cumulative distribution function (CDF) $F(x)$ given by

$$F(x)=F(x;\theta)=1-\exp[-\alpha\lambda_\beta(x)],\ \ x\ge 0,(\alpha,\beta>0). \quad (1.1)$$

where $\theta=(\alpha,\beta)$ and $\lambda_\beta(x)=\lambda(x;\beta)$, is non-negative, continuous, monotone increasing and differentiable function of x such that $\lambda(x;\beta)\to 0$ as $x\to 0^+$ and $\lambda(x;\beta)\to\infty$ as $x\to\infty$.

The probability density function (PDF) of this class is given by

$$f(x)=\alpha\lambda'_\beta(x)\exp[-\alpha\lambda_\beta(x)],\quad x\ge 0.$$

This class of absolutely continuous distributions including, as special cases, Weibull (exponential, Rayleigh as special cases), compound Weibull (or three parameters Burr-type XII), Pareto, power function (beta as a special case), Gompertz and compound Gompertz distributions, among others.

The corresponding reliability function (RF) and the hazard rate function (HRF) are given, respectively by

$$R(x)=\exp[-\alpha\lambda_\beta(x)],\quad x\ge 0,$$
$$h(x)=\alpha\lambda'_\beta(x),\quad x\ge 0.$$

The CDF of finite mixture of two components $F_1(x)$ and $F_2(x)$ from a class (1.1), is given, for $0\le p_1\le 1,$ by

$$F(x)=p_1F_1(x)+p_2F_2(x),$$

where $p_1=p,\ p_2=1-p_1$.

For $q=1,2,$ $F_q(x)$ from (1.1) is,

$$F_q(x)=1-\exp[-\alpha_q\lambda_{\beta_q}(x)],\quad x\ge 0.$$

The PDF of finite mixture $f(x)$ is given by

$$f(x)=p_1f_1(x)+p_2f_2(x), \quad (1.2)$$

where, for $q=1,2,$

$$f_q(x)=\alpha_q\lambda'_{\beta_q}(x)\exp[-\alpha_q\lambda_{\beta_q}(x)],\ x>0,$$

hence the CDF of a finite mixture $F(x)$ of such two components is

$$F(x)=1-p_1\exp[-\alpha_1\lambda_{\beta_1}(x)]-p_2\exp[-\alpha_2\lambda_{\beta_2}(x)].$$

The corresponding RF and HRF are given, respectively, by

$$R(x)=p_1R_1(x)+p_2R_2(x), \quad (1.3)$$
$$H(x)=f(x)/R(x).$$

For the generalized Type-II HCS, the likelihood function can be written, see Chandrasekar *et al.*(2004), when $m\ne -1$ and $m=-1$, respectively, as

$$L(\theta\,|\,x)=\begin{cases} c_{D_1-1}[R(T_1)]^{\gamma_{D_1}-1}\prod_{i=1}^{D_1}[R(x_i)]^m f(x_i),\ D_1=r,\ldots,n, \\ c_{r-1}[R(x_r)]^{\gamma_{r+1}}\prod_{i=1}^{r}[R(x_i)]^m f(x_i),\ D_1=0,1,\ldots,r-1;D_2=r, \\ c_{D_2-1}[R(T_2)]^{\gamma_{D_2}-1}\prod_{i=1}^{D_2}[R(x_i)]^m f(x_i),\ D_2=0,\ldots,r-1, \end{cases} \quad (1.4a)$$

$$L(\theta\mid x)=\begin{cases} k^{D_1}[R(T_1)]^{k-1}\prod_{i=1}^{D_1}H(x_i), & D_1=r,\ldots,n,\\ k^{r}[R(x_r)]^{k-1}\prod_{i=1}^{r}H(x_i), & D_1=0,1,\ldots,r-1;D_2=r,\\ k^{D_2}[R(T_2)]^{k-1}\prod_{i=1}^{D_2}H(x_i), & D_2=0,\ldots,r-1,\end{cases} \tag{1.4b}$$

where $x=(x_1,\ldots,x_r)$, and $C_{t-1}=\prod_{i=1}^{t}\gamma_i,\ \gamma_t=k+(n-t)(m+1).$

It is noticed that, Type-II HCS and Type-I HCS can be obtained as special cases from generalized Type-II HCS as follows:

• Scheme 1: when the experiment terminated at time T_1 with $T_2\to\infty$, we obtain Type-II HCS.

• Scheme 2: when the experiment terminated at time $x_{r:n}$, we obtain two cases

(a) Type-I HCS, when $T_1=0$.

(b) Type-II HCS, when $T_2\to\infty$.

• Scheme 3: when the experiment terminated at time T_2 with $T_1=0$, we obtain Type-I HCS.

We shall use the conjugate prior density, that was suggested by AL-Hussaini (1999b), in the following form

$$\pi(\theta;\nu)\propto C(\theta;\nu)\exp[-D(\theta;\nu)], \tag{1.5}$$

where $\theta=(p,\alpha_1,\alpha_2,\beta_1,\beta_2),\nu\in\Omega$, Ω is the hyperparameter space.

It follows, from (1.4a), (1.4b) and (1.5), that the posterior density function is given by

$$\pi^*(\theta\mid x)=A_1\,C(\theta;\nu)\exp[-D(\theta;\nu)]\,L(\theta\mid x), \tag{1.6}$$

where $A_1^{-1}=\int_\theta \pi(\theta;\nu)L(\theta\mid x)\,d\theta.$

In this paper, Bayesian predictive intervals ($BPI's$) for a future $gos's$ are constructed when the previous (informative) sample is a finite mixture of two components from a general class of continuous distributions under generalized Type-II HCS. Two-sample scheme is used here. In Section 3, illustrative example of finite mixture of two Weibull components is discussed. Specializations are made in $uos's$ and $urv's$ cases. Concluding remarks are presented in Section 4.

2 Bayesian Two Sample Prediction

Suppose that the first r $gos's$ $X_{1;n,m,k},X_{2;n,m,k},\ldots,X_{r;n,m,k},1\le r\le n,$ represents the previous sample of size n from a mixture of two general components from a class of continuous distributions based on generalized hybrid censoring schemes and let $Y_{1;N,M,K},Y_{2;N,M,K},\ldots,$ $Y_{N;N,M,K}$, $K\ge 1$, $M\ge -1$ be a second independent generalized ordered random sample (of size N) of future observations from the same distribution. We wish to predict the future gos $Y_b\equiv Y_{b;N,M,K}$, $b=1,2,\ldots,N,$ in the future sample of size N.

It was shown by Kamps (1995) that the PDF of gos Y_b is given as

$$h^*_{Y_b}(y\mid\theta)\propto\begin{cases} f(y)\sum_{i=0}^{b-1}\omega_i^{(b)}[R(y)]^{\gamma^*_{b-i}-1}, & M\ne -1,\\ {}^{K-1}[\ln R(y)]^{b-1}f(y), & M=-1,\end{cases} \tag{2.1}$$

where $\gamma^*_j=K+(N-j)(M+1)$ and $\omega_i^{(b)}=(-1)^i\binom{b-1}{i}.$

Substituting (1.2) and (1.3) in (2.1), we have

$$h^{*}_{Y_b}(y\mid\theta)\propto\begin{cases}[p_1 f_1(y)+p_2 f_2(y)] \\ \times\sum_{i=0}^{b-1}\omega_i^{(b)}[p_1R_1(y)+p_2R_2(y)]^{\gamma^{*}_{b-i}-1}, & M\neq -1,\\ (\ln[p_1R_1(y)+p_2R_2(y)])^{b-1}[p_1R_1(y)+p_2R_2(y)]^{K-1} \\ \times[p_1 f_1(y)+p_2 f_2(y)], & M=-1.\end{cases} \quad (2.2)$$

The predictive PDF of Y_b, $b=1,2,...,N,$ given the previous $gos's\ x$ is given by

$$f^{*}_{Y_b}(y\mid x)=\int_{\theta}h^{*}_{Y_b}(y\mid\theta)\pi^{*}(\theta\mid x)\,d\theta,\quad y>0. \quad (2.3)$$

Bayesian prediction bounds with caver τ for Y_b, $b=1,2,...,N$ are obtained by evaluating

$$P[Y_b>\nu\mid x]=\int_{\nu}^{\infty}f^{*}_{Y_b}(y\mid x)dy,\quad \nu>0. \quad (2.4)$$

A $100\tau\%$ BPI for Y_b is then given by

$$P[L(x)<Y_b<U(x)]=\tau,$$

where $L(x)$ and $U(x)$ are obtained, respectively, by solving the following two equations

$$P[Y_b>L(x)\mid x]=\frac{1+\tau}{2}, \quad (2.5)$$

$$P[Y_b>U(x)\mid x]=\frac{1-\tau}{2}. \quad (2.6)$$

Since the joint posterior density of the vector parameters θ, $\pi^{*}(\theta\mid x)$, can not be expressed in closed form and hence it can not be evaluated analytically, so we propose to apply Metropolis algorithm to draw MCMC samples. Eberaly and Casella (2003) were interested in the problem of estimating the posterior Bayesian credible region by the MCMC algorithm. Bayarri *et al.*(2006) proposed MCMC algorithms to simulate from conditional predictive distributions.

This technique can be done by rewritten the predictive PDF of Y_b, $b=1,2,...,N$ given the previous $gos's\ x$ as

$$f^{**}_{Y_b}(y\mid x)=\frac{\sum_{i=1}^{\aleph}h^{*}_{Y_b}(y\mid\theta_i,x)}{\sum_{i=1}^{\aleph}\int_0^{\infty}h^{*}_{Y_b}(y\mid\theta_i,x)dy}, \quad (2.7)$$

where $\theta_i, i=1,2,3,...,\aleph$ are generated from the posterior density function (1.6).

A $100\,\tau\%$ BPI (L,U) with cover τ of the future gos Y_b is given by solving the following two nonlinear equations

$$\frac{\sum_{i=1}^{\aleph}\int_L^{\infty}h^{*}_{Y_b}(y\mid\theta_i,x)dy}{\sum_{i=1}^{\aleph}\int_0^{\infty}h^{*}_{Y_b}(y\mid\theta_i,x)dy}=\frac{1+\tau}{2}, \quad (2.8)$$

$$\frac{\sum_{i=1}^{\aleph}\int_U^{\infty}h^{*}_{Y_b}(y\mid\theta_i,x)dy}{\sum_{i=1}^{\aleph}\int_0^{\infty}h^{*}_{Y_b}(y\mid\theta_i,x)dy}=\frac{1-\tau}{2}. \quad (2.9)$$

Numerical methods such as Newton-Raphson, are generally necessary to solve the above two equations to obtain L and U for a given τ.

3 Example (Weibull components)

In this model, for $q=1,2$ and $x>0$, $\lambda_{\beta_q}(x)=x^{\beta_q}$, so $\lambda'_{\beta_q}(x)=\beta_q x^{\beta_q-1}$. So,the q^{th} PDF is

$$f_q(x)=\alpha_q\beta_q x^{\beta_q-1}\exp[-\alpha_q x^{\beta_q}],\quad x>0.$$

Suppose that all parameters are unknown. Let p_1 be independent of α_1,α_2 and independent of β_1,β_2. As a suitable prior distribution of p_1, we consider the beta distribution with positive parameters b_1 and b_2 in the form

$$\pi(p_1)\propto p_1^{b_1-1}p_2^{b_2-1}.$$

Suppose that α_1 and α_2 are distributed as gamma distributions with positive parameters (δ_1,d_1) and (δ_2,d_2), respectively, in the forms

$\pi(\alpha_1)\propto\alpha_1^{\delta_1-1}\exp(-d_1\alpha_1),$ *and* $\pi(\alpha_2)\propto\alpha_2^{\delta_2-1}\exp(-d_2\alpha_2),$ and the prior distributions of β_1 and β_2 are gamma distributions with positive parameters (δ_3,d_3) and

(δ_4, d_4), respectively, in the forms

$\pi(\beta_1) \propto \beta_1^{\delta_3 - 1} \exp(-d_3\beta_1), \quad and \quad \pi(\beta_2) \propto \beta_2^{\delta_4 - 1} \exp(-d_4\beta_2).$

Now, the joint prior density function of $\theta = (p_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$ is given by

$$\pi(\theta) \propto p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \times \exp[-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)]. \tag{3.1}$$

For $uos's$ (previous sample), as special case from the case $m \neq -1$ ($m = 0$ and $k = 1$), by multipling the likelihood function (1.4a) and the prior density function (3.1), the joint posterior density function will be in the form

$$\pi^*(\theta | x) \propto \begin{cases} p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp[-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times [p_1\xi_1(T_1) + p_2\xi_2(T_1)]^{n-D_1} \\ \times \prod_{i=1}^{D_1} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \quad D_1 = r, \ldots, n, \\ \\ p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp[-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times [p_1\xi_1(x_r) + p_2\xi_2(x_r)]^{n-r} \\ \times \prod_{i=1}^{r} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \quad D_1 = 0, 1, \ldots, r-1; D_2 = r, \\ \\ p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp[-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times [p_1\xi_1(T_2) + p_2\xi_2(T_2)]^{n-D_2} \\ \times \prod_{i=1}^{D_2} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \quad D_2 = 0, \ldots, r-1. \end{cases} \tag{3.2}$$

From (3.2), the marginal posterior density of p_1 is

$$\pi^*(p_1 | \alpha_1, \alpha_2, \beta_1, \beta_2, x) \propto \begin{cases} p_1^{b_1-1} p_2^{b_2-1} [p_1 R_1(T_1) + p_2 R_2(T_1)]^{n-D_1} \\ \times \prod_{i=1}^{D_1} [p_1 f_1(x_i) + p_2 f_2(x_i)], \quad D_1 = r, \ldots, n, \\ \\ p_1^{b_1-1} p_2^{b_2-1} [p_1 R_1(x_r) + p_2 R_2(x_r)]^{n-r} \\ \times \prod_{i=1}^{r} [p_1 f_1(x_i) + p_2 f_2(x_i)], \quad D_1 = 0, 1, \ldots, r-1; D_2 = r, \\ \\ p_1^{b_1-1} p_2^{b_2-1} [p_1 R_1(T_2) + p_2 R_2(T_2)]^{n-D_2} \\ \times \prod_{i=1}^{D_2} [p_1 f_1(x_i) + p_2 f_2(x_i)], \quad D_2 = 0, \ldots, r-1. \end{cases} \tag{3.3}$$

Similarly, the marginal posterior densities for α_q and β_q, $q = 1,2$ are given, respectively, by

$$\pi^*(\alpha_q \mid p_1,\beta_1,\beta_2,x) \propto \begin{cases} \alpha_q^{\delta_q-1}\exp[-d_q\alpha_q][p_1R_1(T_1)+p_2R_2(T_1)]^{n-D_1} \\ \times\prod_{i=1}^{D_1}[p_1f_1(x_i)+p_2f_2(x_i)], \qquad D_1=r,\ldots,n, \\ \\ \alpha_q^{\delta_q-1}\exp[-d_q\alpha_q][p_1R_1(x_r)+p_2R_2(x_r)]^{n-r} \\ \times\prod_{i=1}^{r}[p_1f_1(x_i)+p_2f_2(x_i)], \qquad D_1=0,1,\ldots,r-1; D_2=r, \\ \\ \alpha_q^{\delta_q-1}\exp[-d_q\alpha_q][p_1R_1(T_2)+p_2R_2(T_2)]^{n-D_2} \\ \times\prod_{i=1}^{D_2}[p_1f_1(x_i)+p_2f_2(x_i)], \qquad D_2=0,\ldots,r-1, \end{cases} \tag{3.4}$$

$$\pi^*(\beta_q \mid p_1,\alpha_1,\alpha_2,x) \propto \begin{cases} \beta_q^{\delta_s-1}\exp[-d_s\beta_q][p_1R_1(T_1)+p_2R_2(T_1)]^{n-D_1} \\ \times\prod_{i=1}^{D_1}[p_1f_1(x_i)+p_2f_2(x_i)], \qquad D_1=r,\ldots,n, \\ \\ \beta_q^{\delta_s-1}\exp[-d_s\beta_q][p_1R_1(x_r)+p_2R_2(x_r)]^{n-r} \\ \times\prod_{i=1}^{r}[p_1f_1(x_i)+p_2f_2(x_i)], \qquad D_1=0,1,\ldots,r-1; D_2=r, \\ \\ \beta_q^{\delta_s-1}\exp[-d_s\beta_q][p_1R_1(T_2)+p_2R_2(T_2)]^{n-D_2} \\ \times\prod_{i=1}^{D_2}[p_1f_1(x_i)+p_2f_2(x_i)], \qquad D_2=0,\ldots,r-1, \end{cases} \tag{3.5}$$

where $s=3,4$.

Now, we consider the following two special cases:

(1) Both the previous and the future samples are upper order statistics

Here, we consider the future sample is $uos's$, so the predictive PDF (2.7) can be written as

$$f^{**}_{Y_b}(y \mid x) = \frac{\sum_{j=1}^{\aleph} h^*_{1Y_b}(y \mid \theta_j, x)}{\sum_{j=1}^{\aleph}\int_0^\infty h^*_{1Y_b}(y \mid \theta_j, x)dy}, \tag{3.6}$$

where

$h^*_{1Y_b}(y \mid \theta) \propto [p_1f_1(y)+p_2f_2(y)]\sum_{i=0}^{b-1}\omega_i^{(b)}[p_1R_1(y)+p_2R_2(y)]^{(N-b+i)}$,

and $\theta_j=(p_{1j},\alpha_{1j},\alpha_{2j},\beta_{1j},\beta_{2j})$, $j=1,2,3,\ldots,\aleph$ are generated from (3.3), (3.4) and (3.5).

A $100\,\tau\%$ BPI of the future uos Y_b is given by solving the equations

$$\frac{\sum_{j=1}^{\aleph}\int_L^\infty h^*_{1Y_b}(y \mid \theta_j, x)dy}{\sum_{j=1}^{\aleph}\int_0^\infty h^*_{1Y_b}(y \mid \theta_j, x)dy} = \frac{1+\tau}{2}, \tag{3.7}$$

$$\frac{\sum_{j=1}^{\aleph}\int_U^\infty h^*_{1Y_b}(y \mid \theta_j, x)dy}{\sum_{j=1}^{\aleph}\int_0^\infty h^*_{1Y_b}(y \mid \theta_j, x)dy} = \frac{1-\tau}{2}. \tag{3.8}$$

(2) Previous sample is $uos's$ and future sample is $urv's$

Here, we consider the future sample is $urv's$, so the predictive PDF (2.7) can be written as

$$f^{**}_{Y_b}(y \mid x) = \frac{\sum_{j=1}^{\aleph} h^{*}_{2Y_b}(y \mid \theta_j, x)}{\sum_{j=1}^{\aleph} \int_0^{\infty} h^{*}_{2Y_b}(y \mid \theta_j, x)\, dy},$$

where

$h^{*}_{2Y_b}(y \mid \theta) \propto (\ln[p_1 R_1(y) + p_2 R_2(y)])^{b-1} [p_1 f_1(y) + p_2 f_2(y)]$,

and $\theta_j, j = 1,2,...,\aleph$ are generated from (3.3), (3.4) and (3.5).

A $100\,\tau\%$ BPI of Y_b is given by solving the equations

$$\frac{\sum_{j=1}^{\aleph} \int_L^{\infty} h^{*}_{2Y_b}(y \mid \theta_j, x) dy}{\sum_{j=1}^{\aleph} \int_0^{\infty} h^{*}_{2Y_b}(y \mid \theta_j, x) dy} = \frac{1+\tau}{2}, \qquad (3.9)$$

$$\frac{\sum_{j=1}^{\aleph} \int_U^{\infty} h^{*}_{2Y_b}(y \mid \theta_j, x) dy}{\sum_{j=1}^{\aleph} \int_0^{\infty} h^{*}_{2Y_b}(y \mid \theta_j, x) dy} = \frac{1-\tau}{2}. \qquad (3.10)$$

4 Numerical Computations

In this section, 95% $BPI's$ for future $gos's$ independent samples of size $N = 4$ from a mixture of two $Weibull(\alpha_q, \beta_q)$, $q = 1,2$, components based on $uos's$ under generalized Type-II HCS are obtained by considering two sample scheme according to the following steps:

1. For a given values of the prior parameters (b_1, b_2), generate a random value p_1 from the $Beta(b_1, b_2)$ distribution.
2. For a given values of the prior parameters δ_q, d_q for $q = 1,2$, generate a random value α_q from the $Gamma(\delta_q, d_q)$ distribution.
3. For a given values of the prior parameters δ_s, d_s for $s = 3,4$, generate a random value β_q for $q = 1,2$, from the $Gamma(\delta_s, d_s)$ distribution.
4. Using the generated values of p_1, α_1, α_2, β_1 and β_2, we generate ordered sample of size n from a mixture of two $Weibull(\alpha_q, \beta_q)$, $q = 1,2$, components as follows:

- Generate two observations u_1, u_2 from Uniform $(0,1)$.
- if $u_1 \le p_1$, then $x = [-\frac{\ln(1-u_2)}{\alpha_1}]^{\frac{1}{\beta_1}}$, otherwise $x = [-\frac{\ln(1-u_2)}{\alpha_2}]^{\frac{1}{\beta_2}}$.
- Repeat above steps n times to get a sample of size n.

5. The above generated sample was censored using generalized Type-II HCS (and special case from it).
6. Generate ($p_{1j}, \alpha_{1j}, \alpha_{2j}, \beta_{1j}, \beta_{2j}$), $j = 1,2,...,1000$ from (3.3), (3.4), (3.5) using $MCMC$ algorithm.
7. The 95% BPI for the observations from a future independent sample of size $N = 4$ are obtained by solving numerically:

- Eqs. (3.7) and (3.8) with $\tau = 0.95$ in the case that both the previous and the future samples are $uos's$.
- Eqs. (3.9) and (3.10) in the case that the previous sample is $uos's$ and the future sample is $urv's$.

The 95% $BPI's$ for future $gos's$ Y_b, $b = 1,2,3,4$ under generalized Type-II HCS (and special cases from it) are displayed in Tables (1a,b,c) and (2a,b,c). The Number of samples which cover the $BPI's$ is 10000 samples. Numerical results are taking into two different hyper parameters:

Group [1]: $b_1 = 2$, $b_2 = 3$, $d_1 = 3.5$, $d_2 = 2.8$, $d_3 = 1.6$, $d_4 = 0.3$, $\delta_1 = 3.6$, $\delta_2 = 2.5$, $\delta_3 = 2$ and $\delta_4 = 0.4$.

Group [2]: $b_1 = 2$, $b_2 = 3$, $d_1 = 1.5$, $d_2 = 1.8$, $d_3 = 1.6$, $d_4 = 3.3$, $\delta_1 = 3.6$, $\delta_2 = 3.5$, $\delta_3 = 2$ and $\delta_4 = 3.4$.

Table (1a): $BPI's$ of the future $uos\ Y_b$ considering Scheme 1

(n, r) (T_1, T_2)	Y_b	Group [1] (L, U)	Length	CP(%)	Group [2] (L, U)	Length	CP(%)
(15, 9) (0.3, 0.5)	Y_1	(0.00004, 0.4789)	0.47886	95.3	(0.00011, 0.19407)	0.19396	95.1
	Y_2	(0.00150, 1.25919)	1.2576	95.4	(0.00364, 0.41245)	0.40881	95.4
	Y_3	(0.01599, 2.58819)	2.5722	95.6	(0.01860, 0.77554)	0.75694	96.7
	Y_4	(0.06323, 4.97992)	4.91669	97.1	(0.04934, 1.50762)	1.45828	98.4
(33, 20) (0.6, 0.9)	Y_1	(0.00002, 0.33917)	0.33915	95.6	(0.00003, 0.33494)	0.33491	95.1
	Y_2	(0.00133, 0.46158)	0.46025	95.7	(0.00177, 0.45845)	0.45668	95.9
	Y_4	(0.05504, 0.63241)	0.57737	96.8	(0.05414, 0.59213)	0.53799	97.3
	Y_4	(0.15726, 0.86119)	0.70393	98.1	(0.15385, 0.83845)	0.6846	98.4
(50, 44) (0.7, 1)	Y_1	(0.00005, 0.27729)	0.27724	95.2	(0.00003, 0.20068)	0.20065	95.8
	Y_2	(0.00344, 0.40304)	0.3996	94.8	(0.00128, 0.41732)	0.41604	96.4
	Y_3	(0.02303, 0.53513)	0.5121	97.0	(0.01500, 0.58450)	0.56070	97.5
	Y_4	(0.05406, 0.72638)	0.67232	97.9	(0.05464, 0.90036)	0.84572	98.8

T

(n, r) (T_1, T_2)	Y_b	Group [1] (L, U)	Length	CP(%)	Group [2] (L, U)	Length	CP(%)
(15, 9) (0.2, 0.6)	Y_1	(0.00004, 0.38838)	0.38834	95.6	(0.00014, 0.19475)	0.19461	95.4
	Y_2	(0.00176, 0.98669)	0.98493	96.0	(0.00254, 0.42127)	0.41873	96.6
	Y_3	(0.01484, 1.9643)	1.94946	96.8	(0.01423, 0.77610)	0.76187	97.4
	Y_4	(0.05639, 3.6554)	3.59901	97.9	(0.04099, 1.44859)	1.4076	98.3
(33, 30) (0.3, 0.9)	Y_1	(0.00002, 0.33663)	0.33661	95.11	(0.00004, 0.32543)	0.32544	95.6
	Y_2	(0.00134, 0.45308)	0.45174	95.6	(0.00182, 0.44194)	0.44012	95.7
	Y_3	(0.05675, 0.63271)	0.57596	95.8	(0.02908, 0.56957)	0.54049	96.9
	Y_4	(0.16008, 0.81642)	0.65634	96.2	(0.03209, 0.83620)	0.75411	97.7
(50, 47) (0.5, 1)	Y_1	(0.00003, 0.29156)	0.29153	95.9	(0.00230, 0.28110)	0.2788	95.7
	Y_2	(0.00156, 0.42314)	0.42158	96.8	(0.00502, 0.40722)	0.4022	95.9
	Y_3	(0.02405, 0.58489)	0.56084	97.3	(0.02814, 0.53865)	0.51051	97.0
	Y_4	(0.08198, 0.80160)	0.81962	98.21	(0.06074, 0.73009)	0.66935	98.8

T

(n, r) (T_1, T_2)	Y_b	Group [1] (L, U)	Length	CP(%)	Group [2] (L, U)	Length	CP(%)
(15, 9) (0.1, 0.2)	Y_1	(0.00003, 0.69066)	0.69063	95.9	(0.00003, 0.39534)	0.39531	95.91
	Y_2	(0.00156, 2.00203)	2.00047	96.5	(0.00102, 1.01922)	1.0182	96.9
	Y_3	(0.01845, 4.56446)	4.54601	97.1	(0.00933, 2.06134)	2.05201	97.4
	Y_4	(0.06050, 10.2844)	10.2039	97.8	(0.00659, 3.9056)	3.89901	98.9
(33, 30) (0.3, 0.5)	Y_1	(0.00002, 0.32719)	0.32717	95.5	(0.00003, 0.3247)	0.32467	95.0
	Y_2	(0.00793, 0.49564)	0.48771	95.9	(0.00539, 0.46437)	0.45948	95.4
	Y_3	(0.02305, 0.76882)	0.74577	96.7	(0.02351, 0.6285)	0.60499	96.3
	Y_4	(0.09540, 1.70181)	1.60641	98.6	(0.07944, 1.11401)	1.03457	98.9
(50, 47) (0.5, 0.7)	Y_1	(0.01728, 0.27949)	0.26221	95.7	(0.01224, 0.27210)	0.25986	95.4
	Y_2	(0.00160, 0.41153)	0.40993	96.6	(0.00492, 0.40644)	0.40152	95.9
	Y_3	(0.02483, 0.55453)	0.5297	97.1	(0.02628, 0.55064)	0.52436	97.8
	Y_4	(0.05961, 0.77754)	0.71793	97.9	(0.1323, 0.90639)	0.77409	98.45

Table(2a): $BPI^{'}s$ of the future *urv* Y_b considering Scheme 1

(n, r) (T_1, T_2)	Y_b	Group [1] (L, U)	Length	CP(%)	Group [2] (L, U)	Length	CP(%)
(15, 9)	Y_1	(0.00116, 30.1336)	30.13244	98.0	(0.00276, 9.69269)	9.68993	100
(0.3, 0.5)	Y_2	(0.09161, 114.327)	114.23539	97.1	(0.06480, 28.6258)	28.561	99.3
	Y_3	(0.5350, 305.877)	305.342	96.8	(0.23117, 66.7161)	66.48493	98.0
	Y_4	(1.5585, 679.691)	678.1325	96.4	(0.5523, 146.918)	146.3657	97.4
(33, 28)	Y_1	(0.00253, 49.2069)	49.20437	98.8	(0.00548, 11.95361)	11.94813	99.2
(0.6, 0.9)	Y_2	(0.20199, 265.767)	265.56501	98.1	(0.18677, 37.14737)	36.9606	99.0
	Y_3	(0.3757, 693.537)	693.1613	97.5	(0.36056, 76.3252)	75.96464	98.9
	Y_4	(0.5388, 1369.22)	1368.6812	97.1	(0.4787, 229.8767)	229.398	98.0
(50, 44)	Y_1	(0.01182, 53.2567)	53.24488	98.4	(0, 19.0459)	19.0459	99.4
(0.7, 1)	Y_2	(0.1726, 324.956)	324.7834	98.1	(0.0761, 133.788)	133.7119	99.0
	Y_3	(0.3351, 764.899)	764.5639	97.6	(0.2291, 445.35)	445.1209	98.9
	Y_4	(0.5040, 1597.82)	1597.316	97.0	(0.4340, 1104.37)	1103.936	98.7

T

(n, r) (T_1, T_2)	Y_b	Group [1] (L, U)	Length	CP(%)	Group [2] (L, U)	Length	CP(%)
(15, 9)	Y_1	(0.0013, 18.8104)	18.8091	99.0	(0.00183, 9.06734)	9.06551	100
(0.2, 0.6)	Y_2	(0.0806, 65.263)	65.1824	98.6	(0.05549, 29.6474)	29.59191	99.8
	Y_3	(0.4939, 165.665)	165.171	98.1	(0.22866, 62.6221)	62.39344	99.6
	Y_4	(1.4322, 337.073)	335.640	97.9	(0.5337, 108.434)	107.9003	99.4
(33, 30)	Y_1	(0.0004, 18.9266)	18.9262	98.7	(0.00062, 16.7129)	16.71228	98.9
(0.3, 0.9)	Y_2	(0.0349, 81.9796)	81.9447	98.4	(0.03528, 34.7439)	34.70862	98.7
	Y_3	(0.19157, 185.188)	184.9964	98.1	(0.1995, 133.168)	132.9685	98.4
	Y_4	(0.5350, 497.059)	496.524	97.8	(0.5335, 358.595)	358.0615	98.1
(50, 47)	Y_1	(0, 31.0441)	31.0441	99.52	(0.00002, 23.2591)	23.25908	99.7
(0.5, 1)	Y_2	(0.1278, 386.733)	386.605	99.0	(0.1024, 149.639)	149.5366	99.4
	Y_3	(0.295, 1833.00)	1832.70	98.8	(0.2738, 445.361)	445.0872	99.0
	Y_4	(0.466, 5846.52)	5846.05	98.6	(0.4652, 972.607)	972.1418	98.7

T

(n, r) (T_1, T_2)	Y_b	Group [1] (L, U)	Length	CP(%)	Group [2] (L, U)	Length	CP(%)
(15, 9)	Y_1	(0.00152, 46.0456)	46.04408	98.4	(0.0017, 17.5216)	17.5199	99.0
(0.1, 0.2)	Y_2	(0.1183, 172.291)	172.1727	98.0	(0.0889, 52.2267)	52.1378	98.7
	Y_3	(0.778, 433.206)	432.428	97.5	(0.444, 112.526)	112.082	98.3
	Y_4	(2.378, 877.015)	874.637	97.1	(1.1419, 211.369)	210.2271	98.0
(33, 30)	Y_1	(0.0009, 59.06931)	59.06841	99.0	(0.0008, 23.95887)	23.95807	99.8
(0.3, 0.5)	Y_2	(0.0243, 196.5688)	196.5445	99.0	(0.0233, 63.2523)	63.229	99.5
	Y_3	(0.0919, 460.879)	460.7871	99.0	(0.0927, 164.2158)	164.1231	99.0
	Y_4	(0.2096, 951.647)	951.4374	99.0	(0.207, 232.469)	232.262	99.0
(50, 47)	Y_1	(0.0197, 77.28434)	77.26464	99.2	(0.0170, 28.7718)	28.7548	99.5
(0.5, 0.7)	Y_2	(0.1834, 222.3612)	222.1778	98.7	(0.17876, 69.4013)	69.22254	99.1
	Y_3	(0.3501, 544.7461)	544.396	98.3	(0.34837, 171.0893)	170.74093	98.8
	Y_4	(0.520, 975.938)	975.418	98.0	(0.5187, 307.258)	306.7393	98.5

5 Conclusions

1. Bayesian prediction intervals for future observations are obtained using a two-sample scheme based on a finite mixture of two Weibull components model from gos's under generalized Type II HCS. Our results are specialized into two cases:

- Both the previous and the future samples are $uos's$.
- The previous sample is $uos's$ and the future sample is $urv's$.

2. It is evident from Tables (1a,b,c) that, the lengths of the $BPI's$ decrease as the sample size increases. While, from Tables (2a,b,c), the lengths of the $BPI's$ increase as the sample size increases.
3. It is evident from all tables that the lower bounds are relatively insensitive to the specification of the hyper parameters while, the upper bounds are somewhat sensitive.
4. In general, for fixed sample size n and fixed censored sizes r, T_1 and T_2, the length of the $BPI's$ increase by increasing b.

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