On the solution of the Fredholm integral equation with hyper singular kernel

R. T. Matoog

Department of Mathematics, Faculty of Applied Sciences, Umm Al-Qura University, Makkah, Saudi Arabia
r.matoog@yahoo.com

Abstract: in this work, the existence and uniqueness of the solution of Fredholm integral equation of the second kind with hyper singular kernel is presented. The solution of the integral equation, using the orthogonal polynomial method, is discussed in the Chebyshev polynomial form. Moreover, the stability of the numerical solution is considered.

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1.Introduction:

Singular integral equation have received considerable interest the mathematical applications in different areas of sciences for example see Constanda [1], Venturino [2], Kangro [3], Diego [4], and Anastasia [5]. The solution of these problems can be obtained analytically, using the theory developed by Muskhelishvili [6]. The books edited by Popov [7], Tricomi [8], Hocthstab [9] and Green [10] contained many different methods to solve the integral equations analytically. More recently, since analytical method on practical problem often fail, numerical solution of these equations is a much studied subjected of numerous works. The interested reader should consult the fine exposition by Atkinson [11], Delves and Mohamed [12], Golberg [13] and linz [14] for numerical methods

Consider the following **IE**

$$\mu\gamma(x)\phi(x)-\lambda\int_{-1}^{1}\frac{\phi(y)}{(y-x)^2}dy=f(x). \quad (1)$$

Under the conditions

$$\phi(1)=0=\phi(-1) \quad (2)$$

The formula (1) represents a **FIE** of the third kind with hyper **SK** where the solution of the **IE** will be discussed under the condition (2). In (1), μ is a constant defines the kind of the **IE**, and λ is a constant has many physical meaning. The given function $f(x)$ is continuous with its derivatives in the domain of integration. Also, $\gamma(x)$ is a given continuous function. The unknown function $\phi(x), x\in[-1,1]$, will determined under the condition (2). Here, in Eq. (1), the single $\int$ denotes the integration with Cauchy principle value sense.

In order to guarantee the existents of unique solution of Eq. (1), under the condition (2), we assume that the discontinuous kernel in the space of integration satisfies Fredholm condition, the known functions $\gamma(x)$ and $f(x)$ are continuous and the unknown function $\phi(x)$ satisfies the Lipchitz condition. Hence the integral term of (1) exists in the Cauchy principle value sense.

In the remainder part of this work, the **IE** (1) under the condition (2) will be transformed to **FIE** of the second kind with Cauchy kernel. Then, using Chebyshev polynomials of the first and second type with its properties, the solution of **FIE** takes the infinite of linear algebra system equations. Finally, special cases and numerical results are obtained and computed.

1. **Fredholm integral equation**: The **IE** of the third kind with hyper SK of (1) under the condition (2) is equivalent to
2.

$$\mu\gamma(x)\phi(x)-\lambda\int_{-1}^{1}\frac{\frac{d}{dy}\phi(y)}{(y-x)}dy=f(x) \quad (3)$$

The formula (3) represents an integro-differentional equation of the third kind with Cauchy kernel. The importance of (3) in mathematical physics problems came from Frankel [15]. Frankel in his work obtained the solution of (3),

when $\gamma(x)=1$ and $f(x)=-(x-\frac{1}{2}), |x|<1$, by using orthogonal basis functions of the Chebyshev polynomials of the first and second type.

The general solution of (3), using Cauchy method [6], takes the form:

$$\phi(x)=\frac{1}{\lambda\sqrt{1-x^2}}\left\{\int_{-1}^{1}\frac{\sqrt{1-x^2}}{(y-x)}(-\mu\gamma(y)\,\phi(y)+f(x))\,dy+C\lambda\right\} \quad (4)$$

where C is constant will be determined.
Integrating (4) with respect to x and using (2), we have

$$\phi(x)+\frac{\mu}{2\lambda}\int_{-1}^{1}k(x,y)\gamma(y)\phi(y)dy=g(x), \quad (5)$$

$$k(x,y)=2\sqrt{1-y^2}\int\frac{dx}{(y-x)\sqrt{1-x^2}}$$

$$g(x)=\frac{-1}{2\lambda}\int_{-1}^{1}k(x,y)f(y)dy. \quad (6)$$

The formula (5) represents **FIE** of the second kind with discontinuous kernel and discontinuous free term.

The **FIE** (5) has a unique solution in Banach space under the condition:

$$\frac{\mu}{2\lambda}<\|K\|^{-1},\ \|K\|=\left\{\int_{-1}^{1}\int_{-1}^{1}K^2(x,y)\,\gamma(y)\,\gamma(x)\,dydx\right\} \quad (7)$$

Therefore, we can establish that:

i) The **FIE** of the third kind with strong **SK** and continuous free term, under the condition (2) is equivalent to an integro differential equation of the third kind with Cauchy kernel and continues free term. While the both two cases are equivalent to **FIE** of the second kind with Cauchy kernel and discontinuous free term
ii) The formula (4), when $\mu=0$, represents the general solution of Eq. (1) and also, the general solution of Eq. (3).

2. Method of solution:

In spite of the kernel of Eq. (3) has singularity of Cauchy type, the solution of Eq. (3) still smooth with a rapidly convergent Chebyshev expansion. Then, the solution of Eq. (3) represents the same solution of Eq. (1) and Eq. (5).
For this, let

$$T_n(x)=\cos(n\cos^{-1}x),\ x\in[-1,1],\ n\geq 0$$

denotes the Chebyshev polynomials of the first kind, while

$U_n(x)=\dfrac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)},\ n\geq 0$ denotes the Chebyshev polynomials of the second kind. It is well known that $T_n(x)$ from an orthogonal sequence of function with respect to the weight function $(1-x^2)^{-\frac{1}{2}}$, while $U_n(x)$ form an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{\frac{1}{2}}$.

Hence, assume the solution of Eq. (4) in the following form

$$\phi'(x)=\frac{1}{\sqrt{1-x^2}}\sum_{n=0}^{\infty}a_nT_n(x). \quad (8)$$

By integrating (8) with respect to x , we get

$$\phi(x)=\frac{1}{\pi}\left(\frac{\pi}{2}+\sin^{-1}x\right)-\sqrt{1-x^2}\sum_{n=1}^{\infty}a_nT_n(x) \quad (9)$$

Using (8), (9) and the following famous relation [16]

$$\int_{-1}^{1}\frac{T_n(x)dx}{(x-y)\sqrt{1-x^2}}=\pi U_{n-1}(y) \quad (10)$$

The formula (4) becomes

$$\lambda\sum_{n=1}^{\infty}a_nU_{n-1}(x)=\mu\gamma(x)\left[\frac{1}{\pi}\left(\frac{\pi}{2}+\sin^{-1}x\right)-\sqrt{1-x^2}\sum_{n=1}^{\infty}\frac{a_n}{n}U_{n-1}(x)\right]-f(x) \quad (11)$$

Multiplying both sides of (11) by $\sqrt{1-x^2}U_{m-1}(x)dx$, then integrating the result with respect to x from -1 to 1, and then using the famous orthogonal relation [16]

$$\int_{-1}^{1}U_{n-1}(x)U_{m-1}(x)\sqrt{1-x^2}dx=\begin{cases}0, & n\neq m\\ \frac{\pi}{2}, & n=0, n=m=1,2,...,\end{cases} \quad (12)$$

we have the following system of the algebraic equations

$$a_n+\mu'\sum_{m=1}^{\infty}K_{n,m}a_m=C_n,$$

where,

$$\mu'=\frac{\mu}{\lambda},\quad K_{n,m}=\frac{1}{m}R_{n,m}$$

$$R_{n,m}=\frac{2}{\pi}\int_{-1}^{1}U_{n-1}(\zeta)U_{m-1}(\zeta)\sqrt{1-\zeta^2}d\zeta,$$

and

$$C_n = \frac{2}{\lambda\pi}\int_{-1}^{1}\left[\frac{\mu}{\pi}\left(\frac{\pi}{2}+\sin^{-1}x\right)\gamma(x)-f(x)\right]U_{n-1}(x)\sqrt{1-x^2}dx.$$

3. The stability of the infinite system:

To prove that the infinite system of algebraic equations (13) has a unique solution, we consider the space of real bounded sequence, where the metric is defined as

$$\rho(x^{(1)},x^{(2)}) = \sup_{P}\left|x_P^{(1)} - x_P^{(2)}\right| \qquad (14)$$

where, $x = \left\{x_P^{(i)}\right\}_{P=1}^{\infty}$.

Consider an operator

$K : X \to Y = span\left\{U_1, U_2, ..., U_N\right\}$ such that (15)

where, $y = \left\{y_1\right\}_{l=1}^{\infty}$, $x = \left\{x_1\right\}_{l=1}^{\infty}$,

and let $y_1 = c_l - \mu'\sum_{m=1}^{N} K_{l,n} x_n$, $c = \left\{c_1\right\}_{l=1}^{\infty}$, (16)

Moreover, define

$$S_m = \mu'\sum_{n=1}^{N}\left|K_{m,n}\right| = \mu'\sum_{n=1}^{N}\frac{1}{n}\left|R_{m,n}\right| \qquad (17)$$

Applying Cauchy – Munkovisiski inequality, we get

$$S_m \le \mu'\left[\sum_{n=1}^{N}\frac{1}{n^2}\right]^{\frac{1}{2}}\left[\sum_{n=1}^{\infty}R_{m,n}^2\right]^{\frac{1}{2}}$$

Using the formula (17), we obtain

$$S_m \le \mu'\frac{\sqrt{x}}{\sqrt{3}}\left[\int_{-1}^{1}U_{m-1}^2(x)(1-x^2)^{\frac{3}{2}}\gamma^2(x)dx\right]^{\frac{1}{2}},$$

which leads us to obtain

$$S_m \le \frac{\mu}{\lambda}\sqrt{\frac{\pi}{3}}\gamma_0, \quad \gamma_0 = \left[\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}\gamma^2(x)dx\right]^{\frac{1}{2}}. \qquad (18)$$

Hence, the condition of convergence, $S_m \le 1$ leads us to have

$$\mu \le 0.98\lambda\gamma_0. \qquad (19)$$

As special case if $\gamma(x) = 1$, in this case we have the Fredholm integral equation of the second kind, and the eigenvalues of Eq. (13) become

$$K_{n,m} = \frac{2}{n\pi}\int_{-1}^{1}U_{n-1}(\zeta)U_{m-1}(\zeta)(1-\zeta^2)d\zeta \qquad (20)$$

Using the orthogonal relation of Chebechev polynomial of the second kind, we get

$$K_{n,m} = \begin{cases}\dfrac{4m[1+(-1)^{n+m}]}{\pi[n^2-(m-1)^2][n^2-(m+1)^2]} & n\neq m-1,\ n\neq m+1, \\ 0 & n=m-1,\ n=m+1.\end{cases} \qquad (21)$$

Also, we have

$$C_n = \frac{2}{2\pi}\int_{-1}^{1}\left[\frac{\mu}{\pi}\left(\frac{\pi}{2}+\sin^{-1}x\right)-f(x)\right]U_{n-1}(x)\sqrt{1-x^2}dx \qquad (22)$$

Under the stability condition $\mu \le 0.98\lambda$

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