Analytical Expressions of the Jacobi Constants $C_{J_{1,2,3}}$ for the Planar Restricted Three –Body Problem

M.A. Sharaf[1], Z.A.Mominkhan[2]

[1]Department of Astronomy, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Sharaf_adel@hotmail.com
[2]Department of Mathematics, College of Science for Girls King Abdulaziz University, Jeddah, Saudi Arabia
Zammomin @hotmail.com

Abstract: In this paper, analytical power series expressions of the Jacobi constants $C_{J_{1,2,3}}(\mu)$ at the collinear Lagrangian equilibrium points for the planar restricted three – body problem will be established for any desired power of the mass parameter μ.

[Sharaf MA, Mominkhan ZA. **Analytical Expressions of the Jacobi Constants $C_{J_{1,2,3}}$ for the Planar Restricted Three –Body Problem.** *J Am Sci* 2014;10(7):6-9]. (ISSN: 1545-1003). http://www.jofamericanscience.org. 2

Keywords: Three body problem, collinear Lagrangian equilibrium points, expansion theory.

1. Introduction

In the planar restricted three-body problem two bodies (called the primaries) revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction, and a third body attracted by the previous two but not influencing their motion. It is assumed that the third body moves in the plane defined by the two revolving bodies. All the three bodies are considered as point masses. The problem is to find the motion of the third body.

In fact two concepts of the restricted three body problem (R3BP), the combination of which leads to a series of useful research tools, are regularization and Jacobian integral C_J. What concern us in the present paper is Jacobian integral (the only integral of motion of R3BP) .which may be considered one of the most significant features of the qualitative aspects of R3BP. Jacobian integral suggested to Hill (1878) the introduction

of curves of zero velocity for the problem named after him and also for R3BP.Also,it should be reminded of Hill's famous condition regarding the stability of the orbit of the Moon ,which is based entirely on the Jacobian integral. On the other hand, the Jacobian integral plays an important role in the understanding of the totality of possible motions of R3BP(e.g. Marčeta 2012).

Although the countless researches on Jacobian integral and its importance in R3BP as could be detected at once from various Internet sites, no analytical expression of C_J (to the best of the our knowledge) exist. Therefore, the present paper is devoted to establish analytical power series expressions of the Jacobi constants $C_{J_{1,2,3}}(\mu)$ for any desired power of the mass parameter μ.

2. Material and Methods

2-1 Equations of motion

The equations of motion for the planar restricted three –body problem of non-dimensional variables x, y while rotating with the mean motion $n = 1$ are given as,

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}, \tag{1}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} \tag{2}$$

,where

$$2U = C_J \tag{3}$$

,or

$$C_J = x^2 + y^2 + 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right), \tag{4}$$

$$\mu_1 + \mu_2 = 1 \tag{5}$$

$$r_1^2 = (x + \mu_2)^2 + y^2, \tag{6}$$

$$r_2^2 = (x - \mu_1)^2 + y^2, \tag{7}$$

$\mu_{1,2}$ are the masses of the primaries, $r_{1,2}$ are the distance of the third body from the two masses $\mu_{1,2}$ and C_J is the Jacobi constant at the Lagrangian equilibrium point L. Here a dot over a symbol denotes the derivative with respect to the time t.

We can write U in a different form as (Brouwer & Clemence 1961)

$$U = \mu_1\left(\frac{1}{r_1}+\frac{r_1^2}{2}\right)+\mu_2\left(\frac{1}{r_2}+\frac{r_2^2}{2}\right)-\frac{1}{2}\mu_1\mu_2. \tag{8}$$

The advantage of this expression for U is that the explicit dependence on x and y is removed.

2.2 Location of Lagrangian equilibrium points

Due to the gravitational force exerted by the primaries, thesystem has an equilibrium point at the apex of an equilateral triangle with a base formed by the line joining the two mass to be L_4 and the trailing point L_5.There are three more equilibrium points L_1, L_2 and L_3 which lie along the line joining the two masses and are called the *collinear Lagrangian equilibrium points*. The L_1 point lies between the masses μ_1 and μ_2, the L_2 point lies outside the mass μ_2 and the L_3 point lies on the negative x -axis.

To find the location Lagrangian equilibrium points we have to solve the simultaneous nonlinear equations:

$$\frac{\partial U}{\partial x}=\frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial x}+\frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial x}=0, \tag{9}$$

$$\frac{\partial U}{\partial y}=\frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial y}+\frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial y}=0. \tag{10}$$

Using Equations (8) ,(6)and (7),into Equations (9) and (`10) we get the equations for the location of equilibrium points as :

$$\mu_1\left(\frac{-1}{r_1^2}+r_1\right)\frac{x+\mu_2}{r_1}+\mu_2\left(\frac{-1}{r_2^2}+r_2\right)\frac{x-\mu_1}{r_2}=0, \tag{11}$$

$$\mu_1\left(\frac{-1}{r_1^2}+r_1\right)\frac{y}{r_1}+\mu_2\left(\frac{-1}{r_2^2}+r_2\right)\frac{y}{r_2}=0. \tag{12}$$

The locations of L_4 and L_5 when $r_1 = r_2 = 1$, which by Equations (5.4)and (5.5) we get;

$$x=\frac{1}{2}-\mu_2, \quad y=\pm\frac{\sqrt{3}}{2}.$$

Since, $y = 0$ in Equation (12) is the solution of Equation (10),implying that the equilibrium points L_1, L_2 and L_3 lie along the x axis and satisfy Equation (9).

The approximate locations of L_1, L_2 and L_3 are illustrated in the following figures:

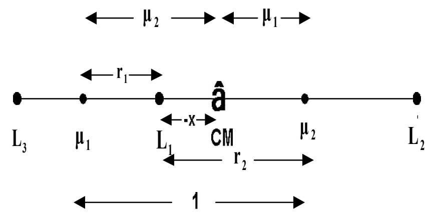

Fig.1: Location of the Lagrangian equilibrium point L_1

From Fig.1 we have

$$r_1+r_2=1, \quad r_1=x+\mu_2, \quad r_2=-x+\mu_1,$$

$$\frac{\partial r_1}{\partial x}=-\frac{\partial r_2}{\partial x}. \tag{13}$$

Hence substituting into Equation (11) we get:

$$\frac{\mu_2}{\mu_1}=3r_2^3\frac{1-r_2+r_2^2/3}{(1+r_2+r_2^2)(1-r_2)^3}. \tag{14}$$

Bošković'

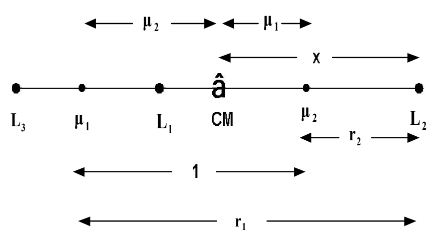

Fig.2: Location of the Lagrangian equilibrium point L_2

From Fig.2 we have

$$r_1-r_2=1, \quad r_1=x+\mu_2, \quad r_2=x-\mu_1,$$

$$\frac{\partial r_1}{\partial x}=\frac{\partial r_2}{\partial x}. \tag{15}$$

Hence substituting for r_1 in Equation (11) we get

$$\frac{\mu_2}{\mu_1}=3r_2^3\frac{1+r_2+r_2^2/3}{(1+r_2)^2(1-r_2^3)}. \tag{16}$$

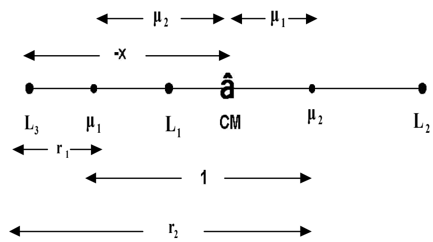

Fig.3: Location of the Lagrangian equilibrium point L_3

From Fig.3 we have

$$r_1 - r_2 = 1, \quad r_1 = -x - \mu_2, \quad r_2 = -x + \mu_1,$$

$$\frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x} = -1. \tag{17}$$

Hence substituting for r_2 in Equation (11) we get

$$\frac{\mu_2}{\mu_1} = \frac{(1-r_1^3)(1+r_1)^2}{r_1^3(r_1^2+3r_1+3)}. \tag{18}$$

Equations (14),(16) and (18) are the basic equations for analytical expressions for $C_{J_{1,2,3}}$.

3. Analytical Expressions of the Jacobi Constants $C_{J_{1,2,3}}$

3.1 Analytical expression of the Jacobi constant C_{J_1}

This expression could be obtained is stepwise fashion as follows: ̃

1- Let $r_2 = \xi$, and $\alpha = \left(\frac{\mu_2}{3\mu_1}\right)^{1/3}$,then Equation (14) becomes

$$\alpha = \frac{\xi(1-\xi+\xi^2/3)^{1/3}}{(1+\xi+\xi^2)^{1/3}(1-\xi)}.$$

2- Expand this equation as power series in ξ up to ξ^{15} (say)

3-Inverting the power series of step 2 using Battin's algorithm [1999] or using *Mathematica* command Inverse Series [ ] to get ξ as power series of α

4-Replace α by $(\mu/3(1-\mu))^{1/3}$, then expand the resulting series in $\mu(=\mu_2)$ to obtain $\xi(=r_2)$ as power series in μ

5- $\mathbf{r_1 = 1 - r_2}$

6- $\mathbf{x = r_1 - \mu}$

7- Expand $C_{J_1} = x^2 + 2\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)$ as power series in $\mu^{1/3}$ we get for the required analytical expression of C_{J_1} the form:

$$C_{J_1}\ \ 3\ \ 3\,3^{1\ 3}\ \ ^{2\ 3}\ \ \frac{10}{3}\ \ \frac{^{4\ 3}}{3\,3^{1\ 3}}\ \ \frac{52\ ^{5\ 3}}{27\,3^{2\ 3}}\ \ \frac{62\ ^{2}}{81}\ \ \frac{46\ ^{7\ 3}}{729\,3^{1\ 3}}$$

$$\frac{2206\ ^{8\ 3}}{6561\,3^{2\ 3}}\ \ \frac{320\ ^{3}}{2187}\ \ \frac{4321\ ^{10\ 3}}{177147\,3^{1\ 3}}\ \ \frac{349052\ ^{11\ 3}}{4782969\,3^{2\ 3}}$$

$$\frac{1552\ ^{4}}{19683}\ \ \frac{7446676\ ^{13\ 3}}{129140163\,3^{1\ 3}}\ \ \frac{29494699\ ^{14\ 3}}{1162261467\,3^{2\ 3}}\ \ \frac{2080\ ^{5}}{59049}$$

3.2 Analytical expression of the Jacobi constant C_{J_2}

1- Let $r_2 = \eta$, and $\alpha = \left(\frac{\mu_2}{3\mu_1}\right)^{1/3}$,then Equation (16) becomes

$$\alpha = \frac{\eta(1+\eta+\eta^2/3)^{1/3}}{(1+\eta)^{2/3}(1-\eta^3)^{1/3}}.$$

2-Apply the corresponding steps of Section 3.1 with $r_1 = 1 + r_2$,we get for the required analytical expression of $C_{j_{21}}$ the form

$$C_{J_2}\ \ 3\ \ 3\,3^{1\ 3}\ \ ^{2\ 3}\ \ \frac{14}{3}\ \ \frac{^{4\ 3}}{3\,3^{1\ 3}}\ \ \frac{56\ ^{5\ 3}}{27\,3^{2\ 3}}\ \ \frac{98\ ^{2}}{81}\ \ \frac{62\ ^{7\ 3}}{729\,3^{1\ 3}}$$

$$\frac{3610\ ^{8\ 3}}{6561\,3^{2\ 3}}\ \ \frac{320\ ^{3}}{2187}\ \ \frac{3023\ ^{10\ 3}}{177147\,3^{1\ 3}}\ \ \frac{1303024\ ^{11\ 3}}{4782969\,3^{2\ 3}}\ \ \frac{2192\ ^{4}}{19683}$$

$$\frac{2128820\ ^{13\ 3}}{129140163\,3^{1\ 3}}\ \ \frac{196243783\ ^{14\ 3}}{1162261467\,3^{2\ 3}}\ \ \frac{5408\ ^{5}}{59049}$$

3.3 Analytical expression of the Jacobi constant C_{J_3}

1-Ler $\alpha = \mu_2/\mu_1$ and $r_1 = 1 + \beta$ then Equation (18) becomes

$$\alpha = -\frac{\beta(2+\beta)^2(3+\beta(3+\beta))}{(1+\beta)^3(7+\beta(5+\beta))}$$

2- Expand this equation as power series in β up to β^{15} (say)

3- As in the above two sections and with $r_1 = 1 + r_2$ and $x = -r_1 - \mu$, we get for the required analytical expression of C_{J_3} the form

$$C_{J_3} \quad 3 \quad \frac{\;}{48}{}^{2} \quad \frac{343}{1728}{}^{3} \quad \frac{5831}{41\,472}{}^{4} \quad \frac{88\,837}{995\,328}{}^{5} \quad \frac{9\,034\,963}{143\,327\,232}{}^{6}$$

$$\frac{6\,840\,449}{143\,327\,232}{}^{7} \quad \frac{516\,193\,391}{13\,759\,414\,272}{}^{8} \quad \frac{271\,061\,766\,563}{8\,916\,100\,448\,256}{}^{9} \quad \frac{10\,800\,497\,146\,439}{427\,972\,821\,516\,288}{}^{10}$$

In concluding the present paper analytical power series expressions of the Jacobi constants $\mathbf{C}_{\mathbf{J}_{1,2,3}}(\mu)$ at the collinear Lagrangian equilibrium points for the planar restricted three –body problem was established for any desired power of the mass parameter μ.

References

1. Battin, R.H. :1999:, *An Introduction to the Mathematics and Methods of Astrodynamics,* Revised Edition, AIAA,Education series, New York.
2. Brouwer,D. & Clemence,C.M: 1961,*Methods of Celestial Mechanics* Acadmaic Press,Inc.New York
3. Hill,G.W.:1878,Am.J.Math.I,5,129
4. Marčeta D: 2012 Publ.Astron. Soc." Rudjer *Boškovic'* "No 11,247

5/2/2014