On Nonlinear Retarded Integral Inequalities of Gronwall Type with an Application

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Abstract: In this paper we establish some new nonlinear integral inequalities of Gronwall-Bellman type. These inequalities generalize some famous inequalities which provide explicit bounds on unknown functions. The inequalities given here can be used as handy tools to study the qualitative as well as quantitative properties of solutions of some nonlinear ordinary differential and integral equations.
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1.Introduction

The integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations.

Lemma1.1:(Gronwall,1919)..Let $x(t)$, $a(t)$ and $f(t)$ be real-valued nonnegative continuous functions defined on $I=[0,\infty)$ with $f(t)\geq 0$.If the inequality

$$x(t)\leq a(t)+\int_0^t f(s)x(s)ds\,,\quad t\in I$$

holds for all $t\in I$,then

$$x(t)\leq a(t)+\int_0^t f(s)a(s)\exp\left(\int_0^s f(\lambda)d\lambda\right),\quad t\in I \tag{1.1}$$

Lemma1.2: (Bellman,1943).Let $x(t)$ and $f(t)$ be real-valued nonnegative continuous functions defined on $I=[0,\infty)$ and x_0 is a constant.If the inequality

$$x(t)\leq x_0+\int_0^t f(s)x(s)ds\,,\quad t\in I$$

holds for all $t\in I$,then

$$x(t)\leq x_0\exp\left(\int_0^t f(s)ds\right),\quad t\in I \tag{1.2}$$

Lemma1.3: (Lipovan,2000).
Let $x,\ f\in C([t_0.T_0],\Re_+)$. Further let $\alpha\in C([t_0.T_0],[t_0,T_0])$ be non decreasing with $\alpha(t)\in t$ on $[t_0,T_0]$, and let k be a nonnegative constant. If the inequality

$$x(t)\leq k+\int_{\alpha(t_0)}^{\alpha(t)} f(s)x(s)ds,\quad t_0<t<T_0$$

Implies that

$$x(t)\leq k\exp\left(\int_{\alpha(t_0)}^{\alpha(t)} f(s)ds\right),\quad t_0<t<T_0 \tag{1.3}$$

Many of the results of Gronwall-Bellman can be cited in [1-11].

2.Main results:

In this section, some new retarded integral inequalities of Gronwall-Bellman type are introduced. Throughout this paper, $\Re$ denotes the set of real numbers, $I=[0,\infty)$, $\Re_+^*=(0,\infty)$, $\Re_1=[1,\infty)$. $C(I,I)$ denotes the set of all nonnegative real-valued continuous functions from I into I and $C^1(I,I)$ denotes the set of all nonnegative real-valued continuously differentiable functions from I into I.

Theorem 2.1: Let $x(t), f(t)$ and $g(t)\in C(I,\Re_+^*)$, $\alpha\in C^1(I,I)$ be nondecreasing with $\alpha(t)\leq t$ on I.If the inequality

$$x(t)\leq x_0+\int_0^{\alpha(t)} f(s)\left[x^{(2-p)}(s)+\int_0^s g(\lambda)x^q(\lambda)d\lambda\right]^p ds, \tag{2.1}$$

holds for all $t \in I$ where $x_0 > 0,\ 0 < p \le 2, 0 \le q < 1$, are constants. Then

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)k_1(\alpha^{-1}(s))ds, \quad \forall t \in I \quad (2.2)$$

where $k_1(t) = \exp\left[p(2-p)\int_0^{\alpha(t)} f(s)ds\right]\left[x_0^{(1-q)(2-p)} + (1-q)\int_0^{\alpha(t)} g(s)\exp\left[-(2-p)(1-q)\int_0^{s} f(\lambda)d\lambda\right]ds\right]^{\frac{p}{1-q}}$, (2.3)

for all $t \in I$.

Proof: Let $M(t)$ be defined as a function by the right-hand side of (2.1). Then

$x(t) \le M(t)$, or $x(\alpha(t)) \le M(\alpha(t)) \le M(t)$ $\forall t \in I$ (2.4)

Differentiating $M(t)$ with respect to t and using (2.4) implies that

$$\frac{dM(t)}{dt} \le \alpha'(t)f(\alpha(t))L^p(t) \quad (2.5).$$

where $L(t) = M^{(2-p)}(t) + \int_0^{\alpha(t)} g(s)M^q(s)ds$,

thus we have $L(0) = M^{(2-p)}(0) = x_0^{(2-p)}$, and

$M(t) \le L(t)$, $\forall t \in I$ (2.6)

Differentiating $L(t)$ with respect to t and using (2.5) and (2.6) leads to

$$\frac{dL(t)}{dt} \le (2-p)\alpha'(t)f(\alpha(t))L(t) + \alpha'(t)g(\alpha(t))L^q(t)$$

(2.7)

Since $L(t) > 0$, then the inequality(2.7) can be rewritten as

$$L^{-q}(t)\frac{dL(t)}{dt} - (2-p)\alpha'(t)f(\alpha(t))L^{(1-q)}(t) \le \alpha'(t)g(\alpha(t))$$

(2.8)

By substituting $z(t) = L^{(1-q)}(t)$, (2.9)

we have $z(0) = L^{(1-q)}(0) \le x_0^{(1-q)(2-p)}$ (2.10)

and $L^{-q}(t)\dfrac{dL(t)}{dt} = \dfrac{1}{(1-q)}\dfrac{dz(t)}{dt}$ then inequality (2.8) takes the form

$$\frac{dz(t)}{dt} - (1-q)(2-p)\alpha'(t)f(\alpha(t))z(t) \le (1-q)\alpha'(t)g(\alpha(t)) \quad (2.11)$$

The inequality (2.11) implies the estimation for $z(t)$ by using (2.10) as

$$z(t) \le \exp\left[(1-q)(2-p)\int_0^{\alpha(t)} f(s)ds\right]\left[x_0^{(1-q)(2-p)} + (1-q)\int_0^{\alpha(t)} g(s)\exp\left[-(2-p)(1-q)\int_0^{s} f(\lambda)d\lambda\right]ds\right]$$

$\forall t \in I$. By using (2.9),the above inequality takes the form

$$L^p(t) \le k_1(t) \quad (2.12)$$

where $k_1(t)$ is defined as in (2.3).By substituting (2.12) in (2.5) we observe that

$$\frac{dM(t)}{dt} \le \alpha'(t)f(\alpha(t))k_1(t) \quad (2.13)$$

By integrating both sides of inequality (2.13) from 0 to $\alpha(t)$ and using (2.6) yields

$$M(t) \le x_0 + \int_0^{\alpha(t)} f(s)k_1(\alpha^{-1}(s))ds, \quad \forall t \in I \quad (2.14)$$

Using (2.14) in (2.4), we get the inequality (2.2).

Remark: If we put $g(t) = 0$, $p = 1$, $x_0 = a(t)$, $q = 1$ and $\alpha(t) = t$ in Theorem2.1 then it reduces to Lemma1.1.

Theorem 2.2: Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \le t$ on I. If the inequality

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)\left[x^p(s) + \int_0^{s} g(\lambda)x^{(2p-1)}(\lambda)d\lambda\right]^p ds,$$

(2.15)

holds for all $t \in I$ where $x_0 > 0,\ p \in (0,1)$ are constants. Then

$$x(t) \le x_0 + \int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(s))ds, \quad \forall t \in I$$

(2.16)

where

$$k_2(t) = \left[x_0^{2p(1-p)} + 2(1-p)\left[p\int_0^{\alpha(t)} f(s)ds + \int_0^{\alpha(t)} g(s)ds\right]\right]^{\frac{p}{2(1-p)}}$$

, (2.17)

for all $t \in I$.

Proof: Let $M(t)$ be defined as a function by the right-hand side of (2.15). Then

$x(t) \le M(t)$, or $x(\alpha(t)) \le M(\alpha(t)) \le M(t)$ $\forall t \in I$ (2.18)

Differentiating $M(t)$ with respect to t and using (2.18) implies that

$$\frac{dM(t)}{dt} \leq \alpha'(t) f(\alpha(t)) L^p(t) \quad (2.19)$$

where $L(t) = M^p(t) + \int_0^{\alpha(t)} g(s) M^{(2p-1)}(s) ds$, thus we have $L(0) = M^p(0) = x_0^p$, and

$M(t) \leq L(t)$, $\forall t \in I$ (2.20)

Differentiating $L(t)$ with respect to t and using (2.19) and (2.20) leads to

$$\frac{dL(t)}{dt} \leq [p\alpha'(t) f(\alpha(t)) + \alpha'(t) g(\alpha(t))] L^{(2p-1)}(t)$$

which can be rewritten as

$$L^{(1-2p)}(t) \frac{dL(t)}{dt} \leq [p\alpha'(t) f(\alpha(t)) + \alpha'(t) g(\alpha(t))]$$

(2.21)

By integrating both sides of inequality (2.21) from 0 to $\alpha(t)$ and using (2.20) yields

$$L^p(t) \leq k_2(t) \quad (2.22)$$

where $k_2(t)$ is defined as in (2.17). By substituting (2.22) in (2.19) we observe that

$$\frac{dM(t)}{dt} \leq \alpha'(t) f(\alpha(t)) k_2(t) \quad (2.23)$$

By integrating both sides of inequality (2.23) from 0 to $\alpha(t)$ and using (2.20) yields

$$M(t) \leq x_0 + \int_0^{\alpha(t)} f(s) k_2(\alpha^{-1}(s)) ds, \quad \forall t \in I \quad (2.24)$$

Using (2.24) in (2.18), we get the inequality (2.16).

Theorem 2.3: Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I. If the inequality

$$x(t) \leq x_0 + \int_0^{\alpha(t)} f(s) \left[x^p(s) + \int_0^s g(\lambda) x^p(\lambda) d\lambda \right] ds,$$

(2.25)

holds for all $t \in I$ where $x_0 > 0,\ 0 < p < 1$ are constants. Then

where

$$x(t) \leq x_0 + \int_0^{\alpha(t)} f(s) \left[x_0^{p(1-p)} + (1-p) \left[p \int_0^s f(\lambda) d\lambda + \int_0^s g(\lambda) d\lambda \right] \right]^{\frac{1}{(1-p)}}$$

(2.26)

for all $t \in I$.

Proof: Let $M(t)$ be defined as a function by the right-hand side of (2.25). Then

$x(t) \leq M(t)$, or $x(\alpha(t)) \leq M(\alpha(t)) \leq M(t)$ $\forall t \in I$ (2.27)

Differentiating $M(t)$ with respect to t and using (2.27) implies that

$$\frac{dM(t)}{dt} \leq \alpha'(t) f(\alpha(t)) L\ (t) \quad (2.28)$$

where $L(t) = M^p(t) + \int_0^{\alpha(t)} g(s) M^p(s) ds$, thus we have $L(0) = M^p(0) = x_0^p$, and

$M(t) \leq L(t)$, $\forall t \in I$ (2.29)

Differentiating $L(t)$ with respect to t and using (2.28) and (2.29) leads to

$$\frac{dL(t)}{dt} \leq [p\alpha'(t) f(\alpha(t)) + \alpha'(t) g(\alpha(t))] L^p(t)$$

which can be rewritten as

$$L^{-p}(t) \frac{dL(t)}{dt} \leq [p\alpha'(t) f(\alpha(t)) + \alpha'(t) g(\alpha(t))]$$

(2.30)

By integrating both sides of inequality (2.30) from 0 to $\alpha(t)$ and using (2.29) yields

$$L(t) \leq \left[x_0^{p(1-p)} + (1-p) \left[p \int_0^s f(\lambda) d\lambda + \int_0^s g(\lambda) d\lambda \right] \right]^{\frac{1}{(1-p)}}$$

(2.31)

By substituting (2.31) in (2.29) and using the fact that $M(0) = x_0$ and by integrating both sides of resulting inequality from 0 to $\alpha(t)$ and using (2.27) also we get the required inequality (2.26).

Theorem 2.4: Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I. If the inequality

$$x(t) \leq x_0 + \int_0^{\alpha(t)} f(s) \left[x^{(2-p)}(s) + \int_0^s g(\lambda) x(\lambda) d\lambda \right]^p ds,$$

(2.32)

holds for all $t \in I$ where $x_0 > 0,\ 0 < p \leq 2$, are constants. Then

$$x(t) \leq x_0 + \int_0^{\alpha(t)} f(s) k_3(\alpha^{-1}(s)) ds, \quad \forall t \in I$$

(2.33)

where

$$k_3(t) = x_0^{p(2-p)}\left[\exp\left[(2-p)\int_0^{\alpha(t)} f(s)ds + \int_0^{\alpha(t)} g(s)ds\right]\right]^p$$

, (2.34)

for all $t \in I$.

Proof: The proof of Theorem 2.4 is the same as the proof of Theorem 2.1 with suitable modifications.

Theorem 2.5:

Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I,I)$ be nondecreasing with $\alpha(t) \le t$ on I and let $n(t) \in C(I,I)$ be nondecreasing. If the inequality

$$x(t) \le n(t) + \int_0^{\alpha(t)} f(s)x^p(s)ds + \int_0^{\alpha(t)} g(s)x(s)ds,$$

(2.35)

holds for all $t \in I$ where $p < 1$, are constants. Then

$$x(t) \le n(t)k_4(t), \quad \forall t \in I \quad (2.36)$$

where

$$k_4(t) = \exp\left[\int_0^{\alpha(t)} g(s)ds\right]\left[1 + (1-p)\int_0^{\alpha(t)} f(s)n^{-(1-p)}(s)\exp\left[-(1-p)\int_0^s g(\lambda)d\lambda\right]ds\right]^{\frac{1}{1-p}}$$

, (2.37)

for all $t \in I$.

Proof: Since $n(t)$ is positive, monotonic nondecreasing function then inequality (2.35) can be written as

$$\left[\frac{x(t)}{n(t)}\right] \le 1 + \int_0^{\alpha(t)} f(s)n^{-(1-p)}(s)\left[\frac{x(s)}{n(s)}\right]^p ds + \int_0^{\alpha(t)} g(s)\left[\frac{x(s)}{n(s)}\right]ds,$$

(2.38)

Let $M(t)$ be defined as a function by the right-hand side of (2.38). Then

$$\frac{x(t)}{n(t)} \le M(t), \text{ or } \frac{x(\alpha(t))}{n(\alpha(t))} \le M(\alpha(t)) \le M(t)$$

$\forall t \in I$ (2.39)

and $M(0) \le 1$ (2.40)

Differentiating $M(t)$ with respect to t and using (2.39) implies that

$$\frac{dM(t)}{dt} \le \alpha'(t)f(\alpha(t))n^{-(1-p)}(\alpha(t))M^p(t) + \alpha'(t)g(\alpha(t))M(t)$$

(2.41)

Since $M(t) > 0$, then the inequality (2.41) takes the form

$$M^{-p}(t)\frac{dM(t)}{dt} - \alpha'(t)g(\alpha(t))M^{(1-p)}(t) \le \alpha'(t)f(\alpha(t))n^{-(1-p)}(\alpha(t))$$

(2.42)

By substituting $z(t) = M^{(1-p)}(t)$, (2.43)

we have

$$z(0) = M^{(1-p)}(0) \le 1 \quad (2.44)$$

and $M^{-p}(t)\dfrac{dM(t)}{dt} = \dfrac{1}{(1-p)}\dfrac{dz(t)}{dt}$ then inequality (2.42) yields

$$\frac{dz(t)}{dt} - (1-p)\alpha'(t)g(\alpha(t))z(t) \le (1-p)\alpha'(t)f(\alpha(t))n^{-(1-p)}(\alpha(t))$$

(2.45)

The inequality (2.45) implies the estimation for $z(t)$ by using (2.44) as

$$z(t) \le \exp\left[(1-p)\int_0^{\alpha(t)} g(s)ds\right]\left[1 + (1-p)\int_0^{\alpha(t)} f(s)n^{-(1-p)}(s)\exp\left[-(1-p)\int_0^s g(\lambda)d\lambda\right]ds\right]^{\frac{1}{1-p}}$$

$\forall t \in I$. By using (2.43),the above inequality takes the form

$$M(t) \le k_4(t) \quad (2.46)$$

where $k_4(t)$ is defined as in (2.37). By substituting (2.46) in (2.39) we observe that

$$x(t) \le n(t)k_4(t)$$

Remark: If we put $f(t) = 0$, $p = 0$, $n(t) = x_0$ and $\alpha(t) = t$ in Theorem2.5 then it reduces to Lemma1.2.

Theorem 2.6: Let $x(t), f(t)$ and $g(t) \in C(I, \Re_+^*)$, $\alpha \in C^1(I,I)$ be nondecreasing with $\alpha(t) \le t$ on I and let $n(t) \in C(I,I)$ be nondecreasing. If the inequality

$$x^p(t) \le n^p(t) + \int_0^{\alpha(t)} f(s)x^p(s)ds + \int_0^{\alpha(t)} g(s)x(s)ds,$$

(2.47)

holds for all $t \in I$ where $p > 1$ is constants. Then

$$x(t) \le n(t)k_5(t), \quad \forall t \in I \quad (2.48)$$

where

$$k_5(t) = \exp\left[\frac{1}{p}\int_0^{\alpha(t)} f(s)ds\right]\left[1 + \frac{(p-1)}{p}\int_0^{\alpha(t)} g(s)n^{-(p-1)}(s)\exp\left[-\frac{(p-1)}{p}\int_0^s f(\lambda)d\lambda\right]ds\right]^{\frac{1}{p-1}}$$

, (2.49)

for all $t \in I$.

Proof: The proof of Theorem2.6 is the same as the proof of Theorem2.5 with suitable modifications.

3.Application: In this section we present an application of the inequality given in Theorem 2.3 to illustrate the usefulness of our results.

Consider the retarded integral equation

$$\begin{cases} \frac{dx(t)}{dt} = M(t, x(\alpha(t)), \int_0^t H(s, x(\alpha(s)))ds, & \forall t \in I \\ x(0) = x_0 \end{cases} \quad (3.1)$$

where $M \in C(I \times \Re^3, \Re), H \in C(I \times \Re, \Re)$, $|x_0| > 0$ is a constant, satisfy the following conditions

$$|H(t, x(t)| \le g(t)|x(t)|^p \quad (3.2)$$

$$\left| M(t, x(\alpha(t)), \int_0^t H(s, x(\alpha(s)))ds \right| \le f(t)\left(|x(t)|^p + \int_0^t |K(s, x(\alpha(s)))| ds \right)$$

(3.3)

$$x_0 + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\left[x_0^{p(1-p)} + (1-p)\int_0^s \left[\frac{pf(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} + \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} \right] d\lambda \right]^{\frac{1}{p}} ds < \infty$$

(3.4)

where f, g, x, p and α are defined as in Theorem 2.3. Integrating both sides of (3.1) from 0 to t, we have

$$x(t) \le x_0 + \int_0^t \left[M(s, x(\alpha(s)), \int_0^s H(\lambda, x(\alpha(\lambda)))d\lambda \right] ds, \ \forall t \in I$$

(3.5)

Using the conditions (3.2) and (3.3) in (3.5), we observe that

$$|x(t)| \le |x_0| + \int_0^t f(s)\left[|x(s)|^p + \int_0^s g(\lambda)|x(\lambda)|^p d\lambda \right] ds,$$

$$|x(t)| \le |x_0| + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\left[|x(s)|^p + \int_0^s \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} |x(\lambda)|^p d\lambda \right] ds$$

(3.6)

Now an suitable application of the inequality given in Theorem 2.3 with modifications to the above inequality leads to

$$|x(t)| \le x_0 + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\left[x_0^{p(1-p)} + (1-p)\int_0^s \left[\frac{pf(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} + \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} \right] d\lambda \right]^{\frac{1}{p}} ds$$

For all.thus from the hypotheses (3.4) and the estimation in (3.6) implies the boundedness of the solution of (3.1).

Competing Interest: The author declares that she has no competing interests.

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