Mixture of exponentiated Frechet distribution based on upper record values

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Abstract: In this paper, we will discuss the mixture distribution consists of two components from exponentiated Frechet distribution (EFD) based on upper record values. We will study the maximum likelihood estimator (MLE) and Bayes estimation under quadratic loss and LINEX loss functions for two parameters θ_1 and θ_2 of distribution, reliability and failure rate functions. Through Monte Carlo simulation, the root mean square errors (RMSEs) of the estimators are computed and compared between them.

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1. Introduction

Mixture models play an important role in many applicable fields, such as medicine, psychology, cluster analysis, life testing, reliability analysis and etc. Mixtures of lifetime distributions occur when two different causes of failure are presented, each with the same parametric form of lifetime distributions. Many authors have studied the finite mixtures of lifetime distributions. Among them are Teicher [32], Titterington *et al.* [33], McLachlan and Basford [23], Lindley [20], McCulloch and Searle [22]. The mixed Weibull distribution as a model for atmospheric data was proposed by Falls [15], who used the method of moments for obtaining the estimators from a complete sample. The maximum likelihood estimation of parameters in mixed Weibull distribution with equal shape parameter from complete and censored Type I sample was considered by Ashour and Jones [9]. Jaheen [16] used the maximum likelihood of mixture distribution. Nassar and Mahmoud [26], Nassar [25] presented statistic of characteristic this models. One of those who were interested in statistical inference about mixtures distribution parameters Rider [30], Al-Hussaini [6]. Chen *et al.* [13] considered the Bayes estimation for mixtures of two Weibull distribution under Type I censoring. They obtained Bayes estimate approximately for mixture distribution consist of two models from Weibull distribution based on Type II censoring. Al-Hussaini *et al.* [7] and Kao [18] studied properties of mixture distribution consisting of two models from Gompertz and parameters estimate by used maximum likelihood and Lindley [19] method for Bayes estimate and John [17] used moment method and maximum likelihood estimate of parameters for mixture distribution consist of two models from gamma. Abu-Zinadah [3] presented mixture consists of k models from exponentiated Pareto distribution for lifetime distribution and found maximum likelihood estimate and Bayes parameters of mixture based on Type II censoring. Bakoban [11] studied two parameters of mixture from exponentiated gamma distribution, reliability and failure rate function by maximum likelihood estimate and Bayes by used Lindley approximately. Badr and Shawky [10] studied two parameters for mixture of exponentiated Frechet distribution.

Record values and the associated statistics are of interest and importance in many areas of real life applications involving data relating to meterology, sport, economics, athletic events, oil, mining surveys and lifetesting. Many authors have studied records and associated statistics. Among them are Ahsanullah [4-5], Resnick [29], Raqab and Ahsanullah [28], Nagaraja [24], Arnold et al. [8], Ragab [27], Abd-Ellah [1-2], Sultan and Balakrishnan [31] and Mahmoud et al. [21] and El-Sagheer [14].

Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed (iid) random variables with cdf F(x) and pdf f(x). Set $Y_n = \max(X_1, X_2, \dots, X_n), n \geq 1$, we say that X_j is an upper record value of this sequence and denoted by $X_{u(j)}$ if $Y_j > Y_{j-1},\ j > 1$. For more detail and references see Nagaraja [24], Ahsanullah [5] and Arnold et al. [8].

In this paper, the basic idea of Lindley [19] approximate form Bayes estimation is used in the case of mixtures of two EFD based on upper record values. The approximate Bayes estimates are obtained and compared with their corresponding maximum likelihood estimates for different sample size.

2. Mixture of k components exponentiated Frechet distribution

Assuming that $T_{u(1)}, T_{u(2)}, \dots, T_{u(n)}$ are the first n upper record values arising from a sequence $\{X_i\}$ of independent and identically distributed (iid) random variables from EFD with pdf as (see, Titterington *et al.* [33]

$$f(t) = \sum_{j=1}^{k} p_j f_j(t), \quad (1)$$

where $f_j(t),\ j = 1,2,\dots,k$ is thej^{th} pdf components from finite mixture distribution with k components from exponentaited Frechet distribution with shape parameter$\alpha = 1$, p_j mixing proportion satisfies the conditionals

$$\sum_{j=1}^{k} p_j = 1,\ \ 0 \le p_j \le 1,$$

$$f_j(t) = \theta_j t^{-2} e^{-t^{-1}} (1 - e^{-t^{-1}})^{\theta_j - 1},\ \ t > 0,\ \ \theta_j > 0. \quad (2)$$

The cumulative distribution function (cdf) of finite mixture with k components from exponentiated Frechet distribution with $\alpha = 1$ as

$$F(t) = \sum_{j=1}^{k} p_j F_j(t), \quad (3)$$

where$F_j(t)$ is thej^{th} cdf components,

$$F_j(t) = 1 - (1 - e^{-t^{-1}})^{\theta_j},\ \ t > 0,\ \ \theta_j > 0. \quad (4)$$

Then, the reliability function of mixture distribution

$$R(t) = 1 - F(t) = \sum_{j=1}^{k} p_j R_j(t), \quad (5)$$

where

$$R_j(t) = (1 - e^{-t^{-1}})^{\theta_j},\ \ t > 0, \theta_j > 0. \quad (6)$$

The failure rate for any distribution is

$$h(t) = \frac{f(t)}{R(t)}, \quad (7)$$

thus, the failure ratefor mixture distribution of k component is

$$h(t) = \sum_{j=1}^{k} p_j h_j(t),\ h_j(t) = \frac{f_j(t)}{R_j(t)}. \quad (8)$$

Now, when $k = 2$, the pdf,cdf, reliability and failure rate functions for finite mixture of two components from EFD, respectively (see, Badr and Shawky [8] are

$$f(t) = p\theta_1 e^{-\frac{1}{t}} t^{-2} (1 - e^{-\frac{1}{t}})^{\theta_1 - 1} + (1-p)\theta_2 e^{-\frac{1}{t}} t^{-2} \left(1 - e^{-\frac{1}{t}}\right)^{\theta_2 - 1}, t > 0,\ \theta_1, \theta_2 > 0, \quad (9)$$

$$F(t) = p\left[1 - \left(1 - e^{-\frac{1}{t}}\right)^{\theta_1}\right] + (1-p)\left[1 - \left(1 - e^{-\frac{1}{t}}\right)^{\theta_2}\right], \quad (10)$$

$$R(t) = p\left(1 - e^{-\frac{1}{t}}\right)^{\theta_1} + (1-\mathrm{p})\left(1 - e^{-\frac{1}{t}}\right)^{\theta_2}, \quad (11)$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{p\theta_1 e^{-\frac{1}{t}} t^{-2} (1-e^{-\frac{1}{t}})^{\theta_1 - 1} + (1-p)p\theta_2 t^{-2} e^{-\frac{1}{t}} (1-e^{-\frac{1}{t}})^{\theta_2 - 1}}{p(1-e^{-\frac{1}{t}})^{\theta_1} + (1-p)(1-e^{-\frac{1}{t}})^{\theta_2}}, \quad (12)$$

where $p_1 = p$ and $p_2 = (1-p)$.

3. Maximum likelihood estimator

Let $T_1, T_2, \dots, T_n$ be a sequence of iid random upper record values from mixture EFD with cdf F(t) and pdf f(t) as

$$f_j(t_i) = \theta_j t_i^{-2} e^{-t_i^{-1}} (1 - e^{-t_i^{-1}})^{\theta_j - 1},\quad t_i > 0,\ \theta_j > 0 \quad (13)$$

$$F_j(t_i) = 1 - \left(1 - e^{-t_i^{-1}}\right)^{\theta_j}, t_i > 0,\ \theta_j > 0,\ \ j = 1, 2. \quad (14)$$

Then, the likelihood function (see, Arnold et al. [8]) can be written as follows

$$L(\boldsymbol{\theta}; \underline{t}) = f(t_n; \boldsymbol{\theta}) \prod_{i=1}^{n-1} \frac{f(t_i; \boldsymbol{\theta})}{R(t_i; \boldsymbol{\theta})}, \quad (15)$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)$, by taking the logarithm of (15), we get

$$l(\boldsymbol{\theta}|\underline{t}) = \log\left[L(\boldsymbol{\theta}|\underline{t})\right] = \sum_{i=1}^{n} \log[f(t_i; \boldsymbol{\theta})] - \sum_{i=1}^{n-1} \log[R(t_i; \boldsymbol{\theta})]. \quad (16)$$

Assuming that the parameters θ_1 and θ_2 are unknown and p is known, the likelihood equations are given by

$$l_j = \frac{\partial l}{\partial \theta_j} = \sum_{i=1}^{n} \left[\frac{1}{f(t_i)} \frac{\partial f_j(t_i)}{\partial \theta_j}\right] - \sum_{i=1}^{n-1} \left[\frac{1}{R(t_i))} \frac{\partial R_j(t_i)}{\partial \theta_j}\right] = 0. \quad (17)$$

Differentiation (13) for θ_j, we get

$$\frac{\partial f_j(t_i)}{\partial \theta_j} = \theta_j {t_i}^{-2} e^{-{t_i}^{-1}} \left(1 - e^{-{t_i}^{-1}}\right)^{\theta_j - 1} \{\log\left[1 - e^{-{t_i}^{-1}}\right] + \theta_j^{-1}\} = f_j(t_i) k_j(t_i), \quad (18)$$

$$k_j(t_i) = \varphi(t_i) + \theta_j^{-1} \text{ and } \varphi(t_i) = \log\left[1 - e^{-{t_i}^{-1}}\right]. \quad (19)$$

Similarly, from (14), we have

$$\frac{\partial R_j(t_i)}{\partial \theta_j} = R_j(t_i) \varphi(t_i). \quad (20)$$

Therefore

$$l_j = \sum_{i=1}^{n} k_j(t_i) \zeta_j(t_i) - \sum_{i=1}^{n-1} \zeta_j^*(t_i) \varphi(t_i),\ j = 1, 2, \quad (21)$$

where $\zeta_j(t_i) = \frac{f_j(t_i)}{f(t_i)}$, $\zeta_j^*(t_i) = \frac{R_j(t_i)}{R(t_i)}$.

The solution of the two nonlinear likelihood equations $l_1 = 0$ and $l_2 = 0$ (from (17)) yields the

MLEs $\hat{\theta}_{1,M}$ and $\hat{\theta}_{2,M}$ for two parameters θ_1, θ_2, respectively. The MLEs of $R(t)$ and $h(t)$ are given, respectively, by (11) and (12) after replacing θ_1 and θ_2 by their corresponding MLEs $\hat{\theta}_{1,M}$and $\hat{\theta}_{2,M}$.

4. Bayes estimation

Let $\Theta_1$and Θ_2 be independent random variables. The joint prior density for random vector $\boldsymbol{\theta} = (\Theta_1, \Theta_2)$, is thus given by

$$g(\boldsymbol{\theta}) = g(\theta_1, \theta_2) = g(\theta_1)g(\theta_2). \qquad (22)$$

Let Θ_j be follows Gamma distribution with shape parameter β_j and scale parameter$\alpha = 1$, i.e. $G(\beta_j, 1)$, the pdf for Θ_j is

$$g_j(\theta_j) = \frac{\theta_j^{\beta_j-1} e^{-\theta_j}}{\Gamma[\beta_j]}, \quad \beta_j > 0, \ \ \theta_j > 0. \qquad (23)$$

Then, the joint prior probability density function for random vector $\boldsymbol{\theta}$ is

$$g(\boldsymbol{\theta}) = \frac{1}{\Gamma[\beta_1]\Gamma[\beta_2]} \theta_1^{\beta_1-1}\theta_2^{\beta_2-1} e^{-(\theta_1+\theta_2)}, \quad \theta_j > 0, \ \beta_j > 0, \ j = 1, 2. \qquad (24)$$

It is well known that the posterior density function of $\boldsymbol{\Theta}$given the observation (data), which is denoted by $q(\boldsymbol{\theta}|\boldsymbol{t})$, is given as

$$q(\boldsymbol{\theta}|\boldsymbol{t}) = \frac{L(\boldsymbol{\theta}|\boldsymbol{t})g(\boldsymbol{\theta})}{\int_\Omega L(\boldsymbol{\theta}|\boldsymbol{t})g(\boldsymbol{\theta})d\boldsymbol{\theta}}, \qquad (25)$$

where $L(\boldsymbol{\theta}|\boldsymbol{t})$ is given by (15), $g(\boldsymbol{\theta})$ by (24) and Ω is the region in the $\boldsymbol{\theta_1\theta_2}$ plane on which the posterior density $q(\boldsymbol{\theta}|\boldsymbol{t})$ is positive.

The Bayes estimator for $\varphi(\boldsymbol{\theta})$, under squared error loss function, is given by

$$\tilde{\varphi} = E[\varphi(\boldsymbol{\theta}|\boldsymbol{T} = \boldsymbol{t})] = \int_\Omega \varphi(\boldsymbol{\theta})q(\boldsymbol{\theta}|\boldsymbol{t})d\boldsymbol{\theta} = \frac{\int_\Omega \varphi(\boldsymbol{\theta})L(\boldsymbol{\theta}|\boldsymbol{t})g(\boldsymbol{\theta})d\boldsymbol{\theta}}{\int_\Omega L(\boldsymbol{\theta}|\boldsymbol{t})g(\boldsymbol{\theta})d\boldsymbol{\theta}}. \qquad (26)$$

The ratio of the integrals in (26) may thus be approximated by using a form due to Lindley [18], which reduces in the case of two parameters, to the form

$$\tilde{\varphi}^* = \varphi^*(\boldsymbol{\theta}) + \frac{S}{2} + \rho_1 S_{12} + \rho_2 S_{21} + \frac{1}{2}[l_{30}^* v_{12} + l_{21}^* c_{12} + l_{12}^* c_{21} + l_{03}^* v_{21}], \qquad (27)$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)$, $S = \sum_{i=1}^2 \sum_{j=1}^2 \varphi_{ij}\sigma_{ij}, \sigma_{ij}$ =(i, j)th element in the matrix Σ ,

$l = \log[L(\boldsymbol{\theta}|\boldsymbol{t})], \Sigma = -[J(\boldsymbol{\theta})]^{-1}, J(\boldsymbol{\theta}) = [l_{ij}]$, $l_{ij} = \frac{\partial^2 l}{\partial\theta_j\partial\theta_i}$. For $i \neq j$,$S_{ij} = \varphi_i\sigma_{ij} + \varphi_j\sigma_{ji}, v_{ij} = (\varphi_i\sigma_{ii} + \varphi_j\sigma_{ij})\sigma_{ii}, c_{ij} = 3\varphi_i\sigma_{ii}\sigma_{ij} + \varphi_j(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2), \rho_i = \frac{\partial\rho}{\partial\theta_i}, \rho = \log[g(\boldsymbol{\theta})], l_{30}^* = \frac{\partial l_{11}}{\partial\theta_1}, \frac{\partial l_{12}}{\partial\theta_1}, l_{12}^* = \frac{\partial l_{12}}{\partial\theta_2}, \ l_{03}^* = \frac{\partial l_{22}}{\partial\theta_2}$. (28)

Now, we apply Lindley's form (27), we first obtain the elements $\sigma_{ij}, i, j = 1{,}2$as follows

$$\sigma_{11} = -\frac{l_{22}}{D}, \sigma_{22} = -\frac{l_{11}}{D}, \sigma_{12} = \sigma_{21} = \frac{l_{12}}{D}, \qquad (29)$$

where

$$D = l_{11}l_{22} - l_{12}^2, \qquad (30)$$

$$l_1 = \frac{\partial l}{\partial\theta_1} = \{\textstyle\sum_{i=1}^n \zeta_1(t_i)k_1(t_i) - \sum_{i=1}^{n-1} \zeta_1^*(t_i)\varphi(t_i)\}, \qquad (31)$$

$$l_2 = \frac{\partial l}{\partial\theta_2} = \{\textstyle\sum_{i=1}^n \zeta_2(t_i)k_2(t_i) - \sum_{i=1}^{n-1} \zeta_2^*(t_i)\varphi(t_i)\} \qquad (32)$$

$$l_{12} = l_{21} = -\{\textstyle\sum_{i=1}^n w(t_i) + \sum_{i=1}^{n-1} \emptyset(t_i)\}, \quad i = 1,2,\ldots,n, \qquad (33)$$

$$w(t_i) = k_1(t_i)k_2(t_i)\zeta_1(t_i)\zeta_2(t_i),$$
$$\emptyset(t_i) = \zeta_1^*(t_i)\zeta_2^*(t_i).$$

Differentiate (21)with respect toθ_j, we get

$$l_{jj} = \frac{\partial l_j}{\partial\theta_j} = \sum_{i=1}^n [\zeta_j(t_i)\grave{k}_j(t_i) + \frac{k_j(t_i)\grave{f}_j(t_i)}{f(t_i)} - \frac{\zeta_j(t_i)k_j(t_i)\grave{f}(t_i)}{f(t_i)} - \textstyle\sum_{i=1}^{n-1}[\frac{\varphi(t_i)\grave{R}_j(t_i)R(t_i)}{R(t_i)} - \frac{\varphi(t_i)R_j(t_i)\grave{R}(t_i)}{(R(t_i))^2}]. \qquad (34)$$

Differentiate (19), (1), (20) and (5) for $\theta_j$we get, respectively,

$$\frac{\partial k_j(t_i)}{\partial\theta_j} = -\theta_j^{-2}, \qquad (35)$$

$$\frac{\partial f(t_i)}{\partial\theta_j} = p_j\frac{\partial f_j(t_i)}{\partial\theta_j} = p_j f_j(t_i)k_j(t_i), \qquad (36)$$

$$\frac{\partial R_j(t_i)}{\partial\theta_j} = R_j(t_i)\varphi(t_i), \qquad (37)$$

$$\frac{\partial R(t_i)}{\partial\theta_j} = p_j R_j(t_i)\varphi(t_i). \qquad (38)$$

By using (35) – (38) and (34), we get

$$l_{jj} = -p_j\left\{\sum_{i=1}^n [\zeta_j(t_i)\theta_j^{-2} - \frac{k_j^2(t_i)f_j(t_i)}{f(t_i)} + \frac{p_j f_j(t_i)\zeta_j(t_i)k_j^2(t_i)}{f(t_i)}\right]$$

$$+\sum_{i=1}^{n-1}[\frac{\varphi^2(t_i)R_j(t_i)}{R(t_i)}-\frac{\varphi^2(t_i)R_j^2(t_i)p_j}{(R(t_i))^2}]\},$$ (39)

$$l_{jj}=-p_j\{\sum_{i=1}^{n}A_j(t_i)+\sum_{i=1}^{n-1}\zeta_j^*(t_i)\varphi^2(t_i)B_j(t_i)\},$$ (40)

for $s=1,2,\ j=1,2$ and $j\neq s$, but $1-p_j\zeta_j(t_i)=p_s\zeta_s(t_i)$,

where

$$A_j(t_i)=\zeta_j(t_i)\theta_j^{-2}-k_j^2(t_i)\zeta_j(t_i)p_s\zeta_s(t_i),\zeta_j(t_i)=\frac{f_j(t_i)}{f(t_i)},\zeta_j^*(t_i)=\frac{R_j(t_i)}{R(t_i)},$$

$$B_j(t_i)=1-p_j\zeta_j^*(t_i),\quad \tau_j(t_i)=\frac{f_j(t_i)}{F(t_i)},\ j=1,2.$$

Then

$$l_{11}=\frac{\partial l_1}{\partial\theta_1}=-p_1\{\sum_{i=1}^{n}A_1(t_i)-\sum_{i=1}^{n-1}\zeta_1^*(t_i)\varphi^2(t_i)B_1(t_i)\},$$ (41)

$$l_{22}=\frac{\partial l_2}{\partial\theta_2}=-p_2\{\sum_{i=1}^{n}A_2(t_i)-\sum_{i=1}^{n-1}\zeta_2^*(t_i)\varphi^2(t_i)B_2(t_i)\},$$ (42)

$$l_{30}^*=\frac{\partial l_{11}}{\partial\theta_1}=-p_1\{\sum_{i=1}^{n}\frac{\partial A_1(t_i)}{\partial\theta_1}-\sum_{i=1}^{n-1}\varphi^2(t_i)\left[\zeta_1^*(t_i)\frac{\partial B_1(t_i)}{\partial\theta_1}+B_1(t_i)\frac{\partial\zeta_1^*(t_i)}{\partial\theta_1}\right],$$ (43)

$$l_{03}^*=\frac{\partial l_{22}}{\partial\theta_2}=-p_2\{\sum_{i=1}^{n}\frac{\partial A_2(t_i)}{\partial\theta_2}-\sum_{i=1}^{n-1}\varphi^2(t_i)\left[\zeta_2^*(t_i)\frac{\partial B_2(t_i)}{\partial\theta_2}+B_2(t_i)\frac{\partial\zeta_2^*(t_i)}{\partial\theta_2}\right].$$ (44)

Differentiate (33) for θ_1, we obtain

$$l_{21}^*=\frac{\partial l_{21}}{\partial\theta_1}=-p_1p_2\{\sum_{i=1}^{n}\frac{\partial w(t_i)}{\partial\theta_1}-\sum_{i=1}^{n-1}\varphi^2(t_i)\frac{\partial\phi(t_i)}{\partial\theta_1}\},$$ (45)

$$l_{12}^*=\frac{\partial l_{12}}{\partial\theta_2}=-p_1p_2\{\sum_{i=1}^{n}\frac{\partial w(t_i)}{\partial\theta_2}-\sum_{i=1}^{n-1}\varphi^2(t_i)\frac{\partial\phi(t_i)}{\partial\theta_2}\},\quad \text{for}\quad j\neq s,\ j,s=1,2,$$ (46)

where

$$\frac{\partial A_j(t_i)}{\partial\theta_j}=\theta_j^{-2}(\frac{\partial\zeta_j(t_i)}{\partial\theta_j})-2\theta_j^{-3}\zeta_j(t_i)+p_s\zeta_s(t_i)\{k_j^2(t_i)(\frac{\partial\zeta_j(t_i)}{\partial\theta_j})+2\zeta_j(t_i)k_j(t_i)\frac{\partial k_j(t_i)}{\partial\theta_j}\},$$

thus

$$\frac{\partial\zeta_j(t_i)}{\partial\theta_j}=\frac{f(t)(\partial f_j(t_i)/\partial\theta_j)-f_j(t_i)(\partial f(t_i)/\partial\theta_j)}{(f(t))^2}=k_j(t_i)\zeta_j(t_i)p_s\zeta_s(t_i),$$

$$\frac{\partial\zeta_j^*(t_i)}{\partial\theta_j}=\varphi(t_i)\zeta_j^*(t_i)B_j(t_i).$$

4.1 Bayes estimation under quadratic loss function

Estimation of two parameters

The two parameters θ_1,θ_2 can be estimate by using Lindley approximation form (27), as follows

Bayes estimation of parameter $\boldsymbol{\theta_1}$

Put $\theta_1=\varphi^*(\theta)$ in (27) for values $i,\ j=1,2$ as follows

$$\varphi_1^*=\frac{\partial\varphi^*}{\partial\theta_1}=1,\qquad \varphi_2^*=\frac{\partial\varphi^*}{\partial\theta_2}=0,$$

$$\varphi_{11}^*=\frac{\partial^2\varphi^*}{\partial\theta_1\partial\theta_1}=0,\varphi_{21}^*=\varphi_{12}^*=0$$

$$S=\varphi_{11}^*\sigma_{11}+\varphi_{12}^*\sigma_{12}+\varphi_{21}^*\sigma_{21}+\varphi_{22}^*\sigma_{22},\quad S=0.$$

By using (28), we get

$S_{12}=\varphi_1^*\sigma_{11}+\varphi_2^*\sigma_{12},\ S_{12}=\sigma_{11},\ S_{21}=\sigma_{12}$, $v_{12}=(\varphi_1^*\sigma_{11}+\varphi_2^*\sigma_{12})\sigma_{11}=\sigma_{11}^2$,

$v_{21}=\sigma_{12}\sigma_{22}$, $c_{12}=3\varphi_1^*\sigma_{11}\sigma_{12}+\varphi_2^*(\sigma_{11}\sigma_{22}+2\sigma_{12}^2)=3\sigma_{11}\sigma_{12}$,

$c_{21}=3\varphi_2^*\sigma_{22}\sigma_{21}+\varphi_1^*(\sigma_{22}\sigma_{11}+2\sigma_{21}^2)=\sigma_{22}\sigma_{11}+2\sigma_{21}^2$,

$$\rho=\ln\left[\frac{1}{\Gamma[\beta_1]\Gamma[\beta_2]}\theta_1^{\beta_1-1}\theta_2^{\beta_2-1}e^{-(\theta_1+\theta_2)}\right],\rho_1=\frac{\partial\rho}{\partial\theta_1}=\frac{(\beta_1-1)}{\theta_1}-1,$$

$$\rho_2=\frac{\partial\rho}{\partial\theta_2}=\frac{(\beta_2-1)}{\theta_2}-1.$$ (47)

By using pervious relations (43) - (47) and (27), we get the Bayes estimator for θ_1 under quadratic loss function, $\hat{\theta}_{1,s}$.

Bayes estimation of parameter $\boldsymbol{\theta_2}$

Put $\theta_2=\varphi^*(\theta)$ in (27) for values $i,j=1,2$, as follows

$$\varphi_1^*=\frac{\partial\varphi^*}{\partial\theta_1}=0,\ \varphi_2^*=\frac{\partial\varphi^*}{\partial\theta_2}=1,\varphi_{11}^*=0,\ \varphi_{22}^*=0,$$

$$\varphi_{12}^*=\varphi_{21}^*=0,$$

$$S=\varphi_{11}^*\sigma_{11}+\varphi_{12}^*\sigma_{12}+\varphi_{21}^*\sigma_{21}+\varphi_{22}^*\sigma_{22},\quad S=0,$$

then

$S_{12}=\varphi_1\sigma_{11}+\varphi_2\sigma_{21}=\sigma_{21},S_{21}=\varphi_2\sigma_{22}+\varphi_1\sigma_{12}=\sigma_{22},v_{12}=\sigma_{12}\sigma_{11},\ v_{21}=\sigma_{22}^2$,

$c_{12}=\sigma_{11}\sigma_{22}+2\sigma_{12}^2,\ c_{21}=3\sigma_{22}\sigma_{21}$.

As before, we can get the Bayes estimator for θ_2 under quadratic loss function, $\hat{\theta}_{2,s}$.

Bayes estimation of reliability function

Put $\varphi^*(\theta) = R(t))$ in (27) for values$i, j = 1,2$, where $R(t)$ defined by (5), then

$$\varphi_1^* = p_1 R_1(t)\varphi(t), \quad \varphi_2^* = p_2 R_2(t)\varphi(t),$$
$$\varphi_{11}^* = p_1 R_1(t)\varphi^2(t),$$
$$\varphi_{22}^* = p_2 R_2(t)\varphi^2(t), \varphi_{21}^* = \varphi_{12}^* = 0, S = \varphi_{11}^*\sigma_{11} + \varphi_{22}^*\sigma_{22},$$

then

$$S_{12} = \varphi_1^*\sigma_{11} + \varphi_2^*\sigma_{21} = [p_1 R_1(t)\varphi(t)]\sigma_{11} + [p_2 R_2(t)\varphi(t)]\sigma_{21},$$

$$S_{21} = [p_2 R_2(t)\varphi(t)]\sigma_{22} + [p_1 R_1(t)\varphi(t)]\sigma_{12},$$
$$v_{ij} = (\varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ij})\sigma_{ii},$$
$$v_{12} = [p_1 R_1(t)\varphi(t)]\sigma_{11}^2 + [p_2 R_2(t)\varphi(t)]\sigma_{12}\sigma_{11},$$
$$v_{21} = [p_2 R_2(t)\varphi(t)]\sigma_{22}^2 + [p_1 R_1(t)\varphi(t)]\sigma_{21}\sigma_{22},$$
$$c_{ij} = 3\varphi_i^*\sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2), c_{12} = 3[p_1 R_1(t)\varphi(t)]\sigma_{11}\sigma_{12} + [p_2 R_2(t)\varphi(t)](\sigma_{11}\sigma_{21} + 2\sigma_{12}^2),$$
$$c_{21} = 3[p_2 R_2(t)\varphi(t)]\sigma_{22}\sigma_{21} + [p_1 R_1(t)\varphi(t)](\sigma_{22}\sigma_{11} + 2\sigma_{21}^2).$$

As before, we can get the Bayes estimator for $R(t)$ under quadratic loss function, $\hat{R}_s$.

Bayes estimation of failure rate function

Put$\varphi^*(\theta) = h(t)$in (27) for values$i, j = 1,2$, where $h(t)$ defined by (7), then

$$\varphi_j^*(t) = \frac{p_j}{(R(t))^2}\{R(t)f_j(t)k_j(t) - f(t)R_j(t)\varphi(t)\}.$$

Then

$$\varphi_{jj}^* = \frac{p_j}{(R(t))^4}[E_1 - E_2],$$

where

$$E_1 = (R(t))^2[R(t)k_j(t)f_j(t)k_j(t) - R(t)f_j(t)\theta_j^{-2} + p_j f_j(t)k_j(t)R_j(t)\varphi(t) - (R(t))^2\varphi(t)[p_j f_j(t)k_j(t) + R_j(t)\varphi(t)],$$
$$E_2 = 2p_j R(t)R_j(t)\varphi(t)\{R(t)f_j(t)k_j(t) - f(t)R_j(t)\varphi(t)\},$$

$f(t), f_j(t), R(t), R_j(t), k_j(t)$are defined in (1), (2), (5), (6) and (19), for values$i, j = 1,2$, we get $\varphi_{ij}^* = \frac{[E_1^* - E_2^*]}{(R(t))^4}$,

$$E_1^* = \frac{1}{(R(t))^2}[\{p_i p_j f_j(t)k_j(t)R_i(t)\varphi(t) - p_i p_j f_i(t)k_i(t)R_j(t)\varphi(t)\},$$
$$E_2^* = -\frac{2}{(R(t))^3}\{p_j f_j(t)k_j(t)R(t) - p_j R_j(t)f(t)\varphi(t)\}p_i R_i(t)\varphi(t)].$$

As before, we can get the Bayes estimator for $h(t)$ under quadratic loss function, $\hat{h}_s$.

4.2 Bayes estimation under LINEX loss function

On the basis of the LINEX loss function, see Zellner [34] and Calabria and Pulcini [12], We define the Bayes estimator under LINEX of $q = q(\boldsymbol{\theta})$, where$\boldsymbol{\theta}$ **is** unknown parameter, as follows

$$\hat{q}_L = -\frac{1}{c}\ln[E(e^{-qc}|\boldsymbol{t})], \quad c \neq 0, \qquad (48)$$

where

$$E(e^{-qc}|\boldsymbol{t}) = \frac{\int_\Omega e^{-cq}L(\boldsymbol{\theta}|\underline{t})g(\boldsymbol{\theta})d\boldsymbol{\theta}}{\int_\Omega L(\boldsymbol{\theta}|\underline{t})g(\boldsymbol{\theta})d\boldsymbol{\theta}}.$$

Let $\varphi^*(\theta) = e^{-cq(\theta)}$, we can use Lindley approximation form (27) for getthe estimators of unknown parameters as follows

Bayes estimation of parameter $\boldsymbol{\theta_1}$

Put $\varphi^*(\boldsymbol{\theta}) = e^{-c\theta_1}$ in (27), we get

$$\varphi_1^* = \frac{\partial\varphi^*}{\partial\theta_1} = -ce^{-c\theta_1}, \ \varphi_2^* = \frac{\partial\varphi^*}{\partial\theta_2} = 0, \ \varphi_{11}^* = c^2e^{-c\theta_1}, \ \varphi_{22}^* = 0,$$ for values $i, j = 1, 2$.

$$\varphi_{ij}^* = \frac{\partial^2\varphi^*}{\partial\theta_i\partial\theta_j} = 0, \ S = 0, \ S_{ij} = \varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ji}, \ S_{12} = -ce^{-c\theta_1}\sigma_{11}, \ S_{21} = -ce^{-c\theta_1}\sigma_{12}.$$

Then

$$v_{ij} = (\varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ij})\sigma_{ii}, v_{12} = -\sigma_{11}^2ce^{-c\theta_1}, \ v_{21} = -\sigma_{12}\sigma_{22}ce^{-c\theta_1},$$
$$c_{ij} = 3\varphi_i^*\sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2), c_{12} = -3\sigma_{11}\sigma_{12}c\, e^{-c\theta_1},$$
$$c_{21} = -(\sigma_{22}\sigma_{11} + 2\sigma_{21}^2)ce^{-c\theta_1}.$$

By using the pervious relations and (48), yield the Bayes estimator under LINEX loss function, $\hat{\theta}_{1,L}$ of θ_1.

Bayes estimation of parameter $\boldsymbol{\theta_2}$

Put $\varphi^*(\boldsymbol{\theta}) = e^{-c\theta_2}$ in (27), we get

$$\varphi_1^* = \frac{\partial\varphi^*}{\partial\theta_1} = 0, \ \varphi_2^* = \frac{\partial\varphi^*}{\partial\theta_2} = -ce^{-c\theta_2}, \varphi_{11}^* = \frac{\partial^2\varphi^*}{\partial\theta_1^2} = 0, \ \varphi_{22}^* = \frac{\partial^2\varphi^*}{\partial\theta_2^2} = c^2e^{-c\theta_2},$$ for values $i, j = 1,2$,

$$\varphi_{ij}^* = \frac{\partial^2\varphi^*}{\partial\theta_1\partial\theta_2} = 0, \quad S = 0, \ S_{ij} = \varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ji}, S_{12} = -ce^{-c\theta_2}\sigma_{21}, \ S_{12} = -ce^{-c\theta_2}\sigma_{22}.$$

Then

$$v_{ij} = (\varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{jj})\sigma_{ii},$$
$$v_{12} = -ce^{-c\theta_2}\sigma_{12}\sigma_{11}, \ v_{21} = -ce^{-c\theta_2}\sigma_{22}^2,$$
$$c_{ij} = 3\varphi_i^*\sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{22}^2), c_{12} = -ce^{-c\theta_2}(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2),$$
$$c_{21} = -3ce^{-c\theta_2}\sigma_{11}\sigma_{12}$$

As before, we can get the Bayes estimator for$\theta_2$under LINEX loss function, $\hat{\theta}_{2,L}$.

Bayes estimation of reliability function

Put $\varphi^*(\boldsymbol{\theta}) = e^{-cR(t)}$ in (27) for values $i, j = 1,2$, where $R(t)$ defined by (5), then

$\varphi_j^* = \frac{\partial \varphi^*}{\partial \theta_j} = -cp_j e^{-cR(t)} R_j(t)\varphi(t)$, $\varphi_{ij}^* = \frac{\partial^2 \varphi^*}{\partial \theta_i \partial \theta_j} = c^2 p_i p_j R_j(t) R_i(t) e^{-cR(t)} (\varphi(t))^2$,

$\varphi_{jj}^* = \frac{\partial^2 \varphi^*}{\partial \theta_j^2} = -cp_j(\varphi(t))^2 R_j(t) e^{-cR(t)} \{p_j R_j(t) + 1\}$, $S_{ij} = \varphi_1^* \sigma_{11} + \varphi_2^* \sigma_{21}$,

$$S_{12} = -ce^{-cR(t)}\varphi(t)[p_1 R_1(t)\sigma_{11} + p_2 R_2(t)\sigma_{21}],$$

$$S_{21} = -ce^{-cR(t)}\varphi(t)[p_2 R_2(t)\sigma_{22} + p_1 R_1(t)\sigma_{12}].$$

Then

$$v_{12} = -ce^{-cR(t)}\varphi(t)[p_1 R_1(t)\sigma_{11}^2 + p_2 R_2(t)\sigma_{12}\sigma_{11}],$$

$$v_{21} = -ce^{-cR(t)}\varphi(t)[p_2 R_2(t)\sigma_{22}^2 + p_1 R_1(t)\sigma_{21}\sigma_{22}],$$

$$c_{ij} = 3\varphi_i^* \sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{22}^2),$$

$$c_{12} = -ce^{-cR(t)}\varphi(t)[3\sigma_{11}\sigma_{12} p_1 R_1(t) + p_2 R_2(t)[\sigma_{11}\sigma_{22} + 2\sigma_{12}^2],$$

$$c_{21} = -ce^{-cR(t)}\varphi(t)[3\sigma_{22}\sigma_{21} p_2 R_2(t) + (\sigma_{22}\sigma_{11} + 2\sigma_{21}^2) p_1 R_1(t)].$$

As before, we can get the Bayes estimator for R(t) under LINEX loss function, $\hat{R}_L(t)$.

Bayes estimation of failure rate function

Put $\varphi^*(\boldsymbol{\theta}) = e^{-ch(t)}$in (27) for values $i, j = 1,2$, where $h(t)$ defined by (7), then

$$\varphi_j^* = \frac{\partial \varphi^*}{\partial \theta_j} = -\frac{cp_j}{(R(t))^2} e^{-ch(t)} \delta_j,$$

where

$$\delta_j = \{R(t) f_j(t) k_j(t) - f(t) R_j(t)\varphi(t)\},$$

$$\varphi_{jj}^* = -\frac{cp_j}{(R(t))^4} e^{-ch(t)} \{(R(t))^2 [p_j f_j(t) R(t)\left(k_j(t)\right)^2 - R(t) f_j(t)\theta_j^{-2}\} + p_j k_j(t) f_j(t) R_j(t)\varphi(t)] - \frac{cp_j \delta_j^2}{(R(t))^2} - 2\,\delta_j p_j R_j(t)\varphi(t)\},$$

$$\varphi_{ij}^* = -cp_i p_j \frac{e^{-ch(t)}}{(R(t))^4} \{(R(t)^2 [\frac{-c\delta_i \delta_j}{(R(t))^2} + f_j(t) k_j(t) R_i(t)\varphi(t) - \varphi(t) R_j(t) f_i(t) k_i(t)] - 2\delta_j R(t) R_i(t)\varphi(t)\},$$

$$S_{ij} = \varphi_i^* \sigma_{ii} + \varphi_j^* \sigma_{ji},$$

$$S_{12} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{p_1\delta_1\sigma_{11} + p_2\delta_2\sigma_{21}\},$$

$S_{21} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{p_2\delta_2\sigma_{22} + p_1\delta_1\sigma_{12}\}$, $v_{ij} = (\varphi_i^*\sigma_{ii} + \varphi_j^*\sigma_{ij})\sigma_{ii}$

$v_{12} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{p_1\delta_1\sigma_{11}^2 + p_2\delta_2\sigma_{12}\sigma_{11}\}$, $v_{12} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{p_2\delta_2\sigma_{22}^2 + p_1\delta_1\sigma_{21}\sigma_{22}\}$,

$$c_{ij} = 3\varphi_i^*\sigma_{ii}\sigma_{ij} + \varphi_j^*(\sigma_{ii}\sigma_{jj} + 2\sigma_{22}^2),$$

$$c_{12} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{3p_1\delta_1\sigma_{11}\sigma_{12} + p_2\delta_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)\},$$

$$c_{21} = -\frac{ce^{-ch(t)}}{(R(t))^2}\{3p_2\delta_2\sigma_{22}\sigma_{21} + p_1\delta_1(\sigma_{22}\sigma_{11} + 2\sigma_{21}^2)\}.$$

As before, we can get the Bayes estimator for $h(t)$ under LINEX loss function, $\hat{h}_L(t)$.

5. Simulation Study

We obtained, in the above sections, MLEs and Bayesian estimates of the vector parameters $\boldsymbol{\theta} = (\theta_1, \theta_2)$, reliability function $R(t)$ and failure rate function $h(t)$ for mixture of EFD with two components. We can obtain Bayes estimation by using quadratic and LINEX loss functions. The MLEs are obtained as well. In order to assess the statistical performance of these estimates, a simulation study is performed for samples of different sizes. We can usethe root meansquare errors(RMSEs) and biases compare between these estimators. The following algorithm will be used to generate the samples of upper record values and then calculate the estimators:

1. Determine optional values of two prior parameters β_1, β_2, then generate two random variables θ_1, θ_2 from Gamma distributions.

2. Generate random sample of size n from uniform distribution$U(0,1)$.

3. Generate random sample of size n from mixture exponentiated Frechet distribution,

$p = 0.5$, $\theta_1, \theta_2$are obtained from step 1, where cdf is obtained in (10).

4. The MLEs of $\hat{\boldsymbol{\theta}} = (\hat{\theta}_{1,M}, \hat{\theta}_{2,M})$of the parameters vector $\boldsymbol{\theta} = (\theta_1, \theta_2)$ obtained by solving, iteratively, the equation

$l_j = \sum_{i=1}^{n}[k_j(t)\zeta_j(t_i)] - \sum_{i=1}^{n-1}[\zeta_j^*(t_i)\varphi(t_i)]$, $i = 1,2, \ldots, n$, $j = 1, 2$.

The estimators $\hat{R}_M(t_0)$ and $\hat{h}_M(t_0)$ of the functions $R(t)$ and h(t) are then computed at some values t_0.

5.The Bayes estimates under quadratic loss function for parameters vector $\hat{\theta}_s = (\hat{\theta}_{1,s}, \hat{\theta}_{2,s})$,

$\hat{R}_s$ and $\hat{h}_s$ by used Lindley approximately defined by (4.2) taking into account the compensation necessary each time to calculate parameters

$\theta = (\theta_1, \theta_2), R(t)$and $h(t)$ which previously contained in section5. We calculate Bayes estimators under LINEX loss function of parameters $\hat{\theta}_L = (\hat{\theta}_{1,L}, \hat{\theta}_{2,L}), \hat{R}_L$ and $\hat{h}_L$ by used the necessary compensation and mentioned in section6 and compensation in

$\hat{\theta}_L = -\frac{1}{c}\ln[E_\theta(e^{-c\theta})]$.

6. The above steps (2-5) are repeated 1000 times, then we calculated biases and the root mean error square root for different sample sizes n. In all above cases the prior parameters

$\beta_1 = 2,\ \beta_2 = 1.5$ which yield the values $\theta_1 = 0.8107, \theta_2 = 0.384095$ are preparing two real values. The true values of R(t) and h(t) when $t = t_0 = 0.75$, are computed to be

$R(0.75) = 0.834717, h(0.75) = 0.784527.$

The root mean square error (first entries) and the biases value (second entries) are displayed in Tables 1-4. The computational results were computed by using $Mathematica$ 9.0.

6. Concluding Remarks

Based on results which obtained in Tables 1-4, we compared between MLEs, Bayes estimators under quadratic and LINEX loss functions for parameters, reliability and failure rate functions for mixture EFD with two components of EFD in different sampling of upper record values. The Bayes estimators are derived in approximate forms using Lindley's method.

Table1: Estimated RMSEs (first entries) and biases(second entries) of different estimators for θ_1

n	$\hat{\theta}_{1,M}$	$\hat{\theta}_{1,s}$	$\hat{\theta}_{1,L}: c = 2.5$
5	0.29473	0.00457	0.02321
	-0.31527	-0.62845	-0.24576
10	0.26178	0.00385	0.01290
	-0.66243	-0.66252	-0.63566
15	0.23452	0.00108	0.01268
	-0.69235	-0.80962	-0.63983
20	0.13698	0.00107	0.01238
	-0.78436	-0.87316	-0.64357

Table2: Estimated RMSEs (first entries) and biases (second entries) of different estimators for θ_2

n	$\hat{\theta}_{2,M}$	$\hat{\theta}_{2,s}$	$\hat{\theta}_{2,L}: c = 2.5$
5	0.02237	.00834	.19188
	-0.91843	-0.81967	-0.89326
10	0.01098	.00807	.01222
	-0.95431	-0.85631	-0.92115
15	0.01095	.00563	.01219
	-0.94852	-0.89564	-0.93522
20	0.01087	.00358	.01208
	-0.96844	-0.94643	-0.95421

Table3: Estimated RMSEs (first entries) and biases (second entries) of different estimators for R(t)

n	$\hat{R}_M(t)$	$\hat{R}_s(t)$	$\hat{R}_L(t): c = 2.5$
5	0.06275	0.60427	0.25154
	-0.18943	-0.25212	-0.59596
10	0.05008	0.57747	0.17986
	-0.22387	-0.27963	-0.60342
15	0.02003	0.43275	0.14953
	-0.44769	-0.49387	-0.62745
20	0.01549	0.41872	0.14485
	-0.45987	-0.53643	-0.66894

Table4: Estimated RMSEs (first entries) and biases (second entries) of different estimators for h(t)

n	$\hat{h}_M(t)$	$\hat{h}_s(t)$	$\hat{h}_L(t): c = 2.5$
5	0.25632	0.12594	0.26053
	-0.44823	-0.28801	-0.74578
10	0.24017	0.12068	0.25976
	-0.55483	-0.44236	-0.75985
15	0.23815	0.01887	0.24473
	-0.60832	-0.56822	-0.76682
20	0.15489	0.01782	0.23986
	-0.68549	-0.66548	-0.77659

Our observations about the results are stated in the following points:

1. In Table 1, the Bayes estimator under quadratic loss function is best from interims value of mean square error root comparison of Bayes estimator under linex loss function and maximum likelihood estimator, respectively, for all value in this table. In terms biased value the best estimator for Bayes estimator under quadratic loss function comparison of the maxi- mum likelihood estimator and Bayes

estimator under linex loss function, respectively, for all value in this table. We note that, from the Table, data decreeing of mean square error root and biased value at increasing for sample size in this Table.2. In Table 2, the Bayes estimator under quadratic loss function is best from interims value of mean square error root comparison of maximum likelihood estimator and under linex loss function, respectively, for all value in this table. In terms biased value the best estimator for the maximum likelihood estimator comparison of under linex loss function and estimator under quadratic loss function, respectively, for all value in this table. We note, from the Table data decreeing mean square error root and biased value at increasing for sample size in this Table.

3. Table3, shows that the MLE is best from interims value of mean square error root comparison of Bayes estimator under LINEX loss function and Bayes estimator under quadratic loss function, respectively, for all value in this table. In terms biased value the best estimator for Bayes estimator under LINEX loss function comparison of Bayes estimator under quadratic loss function, respectively, for all value in this table. We note, from the Table, data decreeing mean square error root and biased value at increasing for sample size.

4. From Table4, the Bayes estimator under quadratic loss function is best from interims value of mean square error root comparison of maximum likelihood estimator and under linex loss function, respectively, for all value in this table. In terms biased value the best estimator for Bayes estimator under LINEX loss function comparison of the maximum likelihood estimator and Bayes estimator under quadratic loss function, respectively, for all value in this table. We note, from the Table data decreeing mean square error root and biased value at increasing for sample size.

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