The Machine Interference Model with Bulk Arrivals and Hyperexponential Service Time Distribution: $M^X/H_r/1/K/N$ with Balking and Reneging

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Abstract: The objective of this paper is to derive the analytical solution of the machine interference model: $M^X/H_r/1/K/N$ with balking and reneging considering the discipline FIFO. Some measures of effectiveness are deduced and some special cases are also obtained.

Keywords: machine interference model; bulk arrival; hyperexponential service; time distribution

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1. Introduction


In the present system, it is assumed that the units arrive at the system in batches of random size $X$, i.e., at each moment of arrival, there is a probability $c_j=Pr(X=j)$ that $j$ units arrive simultaneously, and the interarrival times of batches follow a negative exponential distribution with parameter $\lambda$. It is also assumed that the arriving batches are served in order of their arrival at a service channel consisting of $r$ independent branches with different rates of service (see, Badr [2]).

The service channel is busy if a unit is present in any one of the branches and in case the service channel is busy. The moment the service channel disposes of the unit being served, the unit at the head of the queue, if there be any, enters at random any one of the $r$ branches of the service channel. The probability that it goes to the $i^{th}$ branch is

$$\sigma_i, \sum_{i=1}^{r} \sigma_i = 1.$$  

The service time distribution in the $i^{th}$ branch is negative exponential with rate $\mu$. We assume that we have a finite source (population) of $N$ customers, one server (repairman) is available and the system has finite storage room such that the total number of customers (machines) in the system is no more than $k$. The queue discipline is assumed to be first come, first served (FCFS).

Consider the balk concept with probability:

$$\beta = \text{prob.}\{\text{a unit joins the queue}\},$$  

where $0 \leq \beta < 1$ if $m = 1(1)N$ and $\beta = 1$ if $m = 0$, $m$ is a number of units in the system. It is also assumed that the units may renege according to an exponential distribution,

$$f(t) = \alpha e^{-\alpha t}, t > 0,$$

with parameter $\alpha$. The probability of reneging in a short period of time $\Delta t$ is given by

$$r_m = (m - 1)\alpha \Delta t,$$

for $1 < m \leq N$ and $r_m = 0$, for $m = 0, 1$.

The steady-state equations and their solution

Let $P_{m,i}$ denote the equilibrium probability that there are $m$ units in the system and the unit in service occupies $i^{th}$ branch, $m = 1(1)k$, $i = 1(1)r$.

$P_0$ denote the equilibrium probability that there are no units in the system.

Then, the steady-state probability difference equations, in the presence of balking and reneging, are:
\[
N \lambda P_0 = \sum_{s=1}^{r} \mu_s P_{1,s}, \quad \text{m=0} \quad (1)
\]
\[
[(N - 1) \beta \lambda + \mu_i] P_{1,i} = \sigma_i \sum_{s=1}^{r} (\mu_s + \alpha) P_{2,s,i} + N \lambda \sigma_i c_i P_0, \quad \text{m=1} \quad (2)
\]
\[
[(N - m) \beta \lambda + \mu_i + (m - 1) \alpha] P_{m+1,i} = \sigma_i \sum_{s=1}^{r} (\mu_s + m \alpha) P_{m+1,s,i} + N \lambda \sigma_i c_i P_0, \quad \text{m=2(1)k-1} \quad (3)
\]
\[
[(N - k) \beta \lambda + \mu_i + (k - 1) \alpha] P_{k,i} = \beta \lambda \sum_{j=1}^{k-1} (N - j) c_j P_{k-j,i} + N \lambda \sigma_i c_k P_0, \quad \text{m=k} \quad (4)
\]

where \(i = 1(1)r\).

Summing (2) over \(i\) and using (1),
\[
\sum_{i=1}^{r} (\mu_i + \alpha) P_{2,i} = (N - 1) \beta \lambda \sum_{i=1}^{r} P_{1,i} + N \lambda (1 - c_i) P_0. \quad (5)
\]

Also, summing (3) over \(i\) and using (5),
\[
\sum_{i=1}^{r} (\mu_i + m \alpha) P_{m+1,i} = \beta \lambda \sum_{j=1}^{m} \sum_{i=1}^{r} (N - j) P_{j,i} - \beta \lambda \sum_{n=1}^{m-1} \sum_{j=1}^{r} \sum_{i=1}^{r} (N - j) c_n P_{j,i} - N \lambda P_0 \sum_{i=1}^{r} c_i + N \lambda P_0, \quad \text{m=2(1)k-1} \quad (6)
\]

From (2) and (5),
\[
[B(1,i) - (N - 1) \beta \lambda \sigma_i] P_{1,i} - (N - 1) \beta \lambda \sigma_i \sum_{s \neq i}^{r} P_{s,i} = N \lambda \sigma_i P_0, \quad \text{i=1(1)r}, \quad (7)
\]

where
\[
B(m,i) = (N - m) \beta \lambda + \mu_i + (m - 1) \alpha, \quad \text{m=1(1)k, i=1(1)r},
\]

which can be written in the matrix form as
\[
B_1 P_1 = N \lambda P_0 S,
\]

where
\[
B_1 = \begin{bmatrix} b_{y}(m) \end{bmatrix},
\]

such that
\[
b_{y}(m) = -(N - m) \beta \lambda \sigma_i, \quad i \neq j
\]
\[
b_{y}(m) = B(m,i) - (N - m)) \beta \lambda \sigma_i
\]

\[
P^T_m = \left[ P_{m,1}, P_{m,2}, ..., P_{m,r} \right], \quad m = 1(1)k - 1
\]

and
\[
S^T = [\sigma_1, \sigma_2, ..., \sigma_r]
\]

where \(T\) denotes the transpose of a matrix.

Now, the inverse matrix of \(B_1\) is given by
\[
B_1^{-1} = [b_{y}^{*}(1)],
\]
where
\[ b^*_j(m) = \frac{(N - m)\beta\lambda\sigma_i}{B(m,i)B(m,j)D_m}, \quad i \neq j \]
\[ b^*_n(m) = \frac{1}{B(m,i)} + \frac{(N - m)\beta\lambda\sigma_i}{B^2(m,i)D_m}, \]
such that
\[ D_m = 1 - (N - m)\beta\lambda \sum_{i=1}^r \frac{\sigma_i}{B(m,i)}, \quad m = 1(1)k. \]

Using this value of \( B^{-1} \) in (8), we have
\[ P_{i,j} = \frac{N\lambda\sigma_i}{B(1,i)D_1}, \quad i=1(1)r. \]  
Similarly, from (3) and (6),
\[ [B(m,i) - (N - m)\beta\lambda\sigma_i] P_{m,i} - (N - m)\beta\lambda\sigma_i \sum_{n=1}^r P_{m,s} = \beta\lambda \sum_{j=1}^{m-1} (N - m + j)\mu_j \sum_{n=1}^r P_{m-s,n} - R(m)\sigma_i \]
where
\[ R(m) = -\beta\lambda \sum_{j=1}^{m-1} (N - j)\mu_j P_{m-j,1} + \beta\lambda \sum_{n=1}^r \sum_{s=1}^r (N - j)\mu_n P_{m-j,s} + N\lambda P_0 \sum_{j=1}^{m-1} \mu_j = N\lambda P_0, \]
which can be written in the matrix form as:
\[ B_m P_m = E - R(m) S \]
where
\[ B_m = [b_{ij}(m)], \quad m = 2(1)k-1 \]
\[ E^T = [\beta\lambda \sum_{j=1}^{m-1} (N - j)\mu_j P_{m-j,1}, \beta\lambda \sum_{j=1}^{m-1} (N - j)\mu_j P_{m-j,2}, \ldots, \beta\lambda \sum_{j=1}^{m-1} (N - j)\mu_j P_{m-j,r}]. \]

Now, the inverse matrix of \( B_m \) is given by
\[ B^{-1}_m = [b^*_0(m)]. \]

Using this value of \( B^{-1}_m \) in (10), we get
\[ P_{m,i} = \frac{\beta\lambda}{B(m,i)} \left\{ \sum_{j=1}^{m-1} (N - m + j)\mu_j P_{m-j,i} + \frac{(N - m)\beta\lambda\sigma_i}{D_m} \sum_{s=1}^r \sum_{j=1}^{m-1} (N - m + j)\mu_j P_{m-j,s} \right\} \]
\[ - \frac{R(m)\sigma_i}{B(m,i)D_m}, \quad i=1(1)r, \quad m = 2(1)k-1. \]

From (4),
\[ P_{k,i} = \frac{\beta\lambda}{\mu_i + (k - 1)\alpha} \left\{ \beta\lambda \sum_{j=1}^{k-1} (N - k + j)\mu_j P_{k-j,i} + N\lambda c_k \sigma_i P_0 + \beta\lambda \sum_{j=2}^k \sum_{n=k-j+1}^r (N - n)\mu_n P_{n,i} \right\}, \]
\[ m = k. \]

Equations (9), (11) and (12) are the required recurrence relations, that give all the probabilities in terms of \( P_0 \), which itself may now be determined by using the normalizing condition:
\[ P_0 + \sum_{m=1}^{k} \sum_{s=1}^{r} P_{m,s} = 1 \]

hence all the probabilities are completely known in terms of the queue parameters.

The following example illustrates the method discussed above.

**Example:**

In the above model: \( M^V/H/1/k/N \) with balking and reneging, letting \( r = 3, k=5 \) and \( N = 8 \), i.e., the model: \( M^V/H/1/5/8 \) with balking and reneging, the results are:

\[
\begin{align*}
P_{1,1} &= a_1 P_0, \quad P_{1,2} = a_2 P_0, \quad P_{1,3} = a_3 P_0, \quad P_{2,1} = b_1 P_0, \quad P_{2,2} = b_2 P_0, \\
P_{2,3} &= b_3 P_0, \quad P_{3,1} = d_1 P_0, \quad P_{3,2} = d_2 P_0, \quad P_{3,3} = d_3 P_0, \quad P_{4,1} = e_1 P_0, \\
P_{4,2} &= e_2 P_0, \quad P_{4,3} = e_3 P_0, \quad P_{5,1} = f_1 P_0, \quad P_{5,2} = f_2 P_0, \quad P_{5,3} = f_3 P_0, \\
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= \frac{8\lambda \sigma_1}{(7\beta \lambda + \mu_1)D_1}, \quad a_2 = \frac{8\lambda \sigma_2}{(7\beta \lambda + \mu_2)D_1}, \quad a_3 = \frac{8\lambda \sigma_3}{(7\beta \lambda + \mu_3)D_1}, \\
b_1 &= \frac{\beta \lambda}{6\beta \lambda + \mu_1 + \alpha} \left\{ 7c_1 a_1 + \frac{42\beta \lambda c_1 \sigma_1}{D_2} \left[ \frac{a_1}{6\beta \lambda + \mu_1 + \alpha} + \frac{a_2}{6\beta \lambda + \mu_2 + \alpha} + \frac{a_3}{6\beta \lambda + \mu_3 + \alpha} \right] \right\}, \\
b_2 &= \frac{\beta \lambda}{6\beta \lambda + \mu_2 + \alpha} \left\{ 7c_1 a_2 + \frac{42\beta \lambda c_2 \sigma_2}{D_2} \left[ \frac{a_1}{6\beta \lambda + \mu_1 + \alpha} + \frac{a_2}{6\beta \lambda + \mu_2 + \alpha} + \frac{a_3}{6\beta \lambda + \mu_3 + \alpha} \right] \right\}, \\
b_3 &= \frac{\beta \lambda}{6\beta \lambda + \mu_3 + \alpha} \left\{ 7c_1 a_3 + \frac{42\beta \lambda c_3 \sigma_3}{D_2} \left[ \frac{a_1}{6\beta \lambda + \mu_1 + \alpha} + \frac{a_2}{6\beta \lambda + \mu_2 + \alpha} + \frac{a_3}{6\beta \lambda + \mu_3 + \alpha} \right] \right\}, \\
\end{align*}
\]

\[
\begin{align*}
d_1 &= \frac{\beta \lambda}{5\beta \lambda + \mu_1 + 2\alpha} \left\{ 6c_1 b_1 + 7c_2 a_1 + \frac{5\beta \lambda \sigma_1}{D_3} \left[ \frac{6c_1 b_1 + 7c_2 a_1}{5\beta \lambda + \mu_1 + 2\alpha} + \frac{6c_1 b_2 + 7c_2 a_2}{5\beta \lambda + \mu_2 + 2\alpha} + \frac{6c_1 b_3 + 7c_2 a_3}{5\beta \lambda + \mu_3 + 2\alpha} \right] \right\}, \\
d_2 &= \frac{\beta \lambda}{5\beta \lambda + \mu_2 + 2\alpha} \left\{ 6c_1 b_2 + 7c_2 a_2 + \frac{5\beta \lambda \sigma_2}{D_3} \left[ \frac{6c_1 b_1 + 7c_2 a_1}{5\beta \lambda + \mu_1 + 2\alpha} + \frac{6c_1 b_2 + 7c_2 a_2}{5\beta \lambda + \mu_2 + 2\alpha} + \frac{6c_1 b_3 + 7c_2 a_3}{5\beta \lambda + \mu_3 + 2\alpha} \right] \right\}, \\
d_3 &= \frac{\beta \lambda}{5\beta \lambda + \mu_3 + 2\alpha} \left\{ 6c_1 b_3 + 7c_2 a_3 + \frac{5\beta \lambda \sigma_3}{D_3} \left[ \frac{6c_1 b_1 + 7c_2 a_1}{5\beta \lambda + \mu_1 + 2\alpha} + \frac{6c_1 b_2 + 7c_2 a_2}{5\beta \lambda + \mu_2 + 2\alpha} + \frac{6c_1 b_3 + 7c_2 a_3}{5\beta \lambda + \mu_3 + 2\alpha} \right] \right\}, \\
\end{align*}
\]
\[ -\sigma_3 \frac{7\beta\lambda(a_1 + a_2 + a_3)(c_1 + c_2 - 1) + 6\beta\lambda(b_1 + b_2 + b_3)(c_1 - 1) + 8\lambda(c_1 + c_2 - 1)}{(5\beta\lambda + \mu_1 + 2\alpha)D_1} \]

\[ e_1 = \frac{\beta\lambda}{4\beta\lambda + \mu_1 + 3\alpha} \left\{ 5c_1d_1 + 6c_2b_1 + 7c_3a_1 + \frac{4\beta\lambda\sigma_1}{D_1} \right\} - \frac{Y\sigma_1}{(4\beta\lambda + \mu_1 + 3\alpha)D_4} \]

\[ + \frac{5c_1d_1 + 6c_2b_1 + 7c_3a_1}{4\beta\lambda + \mu_1 + 3\alpha} \left\{ 5c_1d_1 + 6c_2b_1 + 7c_3a_1 + \frac{4\beta\lambda\sigma_1}{D_4} \right\} - \frac{Y\sigma_1}{(4\beta\lambda + \mu_1 + 3\alpha)D_4} \]

\[ e_2 = \frac{\beta\lambda}{4\beta\lambda + \mu_1 + 3\alpha} \left\{ 5c_1d_2 + 6c_2b_2 + 7c_3a_2 + \frac{4\beta\lambda\sigma_2}{D_4} \right\} - \frac{Y\sigma_2}{(4\beta\lambda + \mu_1 + 3\alpha)D_4} \]

\[ + \frac{5c_1d_2 + 6c_2b_2 + 7c_3a_2}{4\beta\lambda + \mu_1 + 3\alpha} \left\{ 5c_1d_2 + 6c_2b_2 + 7c_3a_2 + \frac{4\beta\lambda\sigma_2}{D_4} \right\} - \frac{Y\sigma_2}{(4\beta\lambda + \mu_1 + 3\alpha)D_4} \]

\[ e_3 = \frac{\beta\lambda}{4\beta\lambda + \mu_1 + 3\alpha} \left\{ 5c_1d_3 + 6c_2b_3 + 7c_3a_3 + \frac{4\beta\lambda\sigma_3}{D_4} \right\} - \frac{Y\sigma_3}{(4\beta\lambda + \mu_1 + 3\alpha)D_4} \]

\[ + \frac{5c_1d_3 + 6c_2b_3 + 7c_3a_3}{4\beta\lambda + \mu_1 + 3\alpha} \left\{ 5c_1d_3 + 6c_2b_3 + 7c_3a_3 + \frac{4\beta\lambda\sigma_3}{D_4} \right\} - \frac{Y\sigma_3}{(4\beta\lambda + \mu_1 + 3\alpha)D_4} \]

\[ f_1 = \frac{1}{\mu_1 + 4\alpha} \left\{ \beta\lambda[4e_1 + 5d_1(1-c_1) + 6b_1(c_1 + c_4 + c_3) + 7a_1(c_4 + c_3)] + 8\lambda c_3 \sigma_1 \right\} \]

\[ f_2 = \frac{1}{\mu_2 + 4\alpha} \left\{ \beta\lambda[4e_2 + 5d_2(1-c_1) + 6b_2(c_3 + c_4 + c_3) + 7a_2(c_4 + c_3)] + 8\lambda c_3 \sigma_2 \right\} \]

\[ f_3 = \frac{1}{\mu_3 + 4\alpha} \left\{ \beta\lambda[4e_3 + 5d_3(1-c_1) + 6b_3(c_3 + c_4 + c_3) + 7a_3(c_4 + c_3)] + 8\lambda c_3 \sigma_3 \right\} \]

\[ D_1 = 1 - 7\beta\lambda \left[ \frac{\sigma_1}{7\beta\lambda + \mu_1} + \frac{\sigma_2}{7\beta\lambda + \mu_2} + \frac{\sigma_3}{7\beta\lambda + \mu_3} \right] \]

\[ D_2 = 1 - 6\beta\lambda \left[ \frac{\sigma_1}{6\beta\lambda + \mu_1 + \alpha} + \frac{\sigma_2}{6\beta\lambda + \mu_2 + \alpha} + \frac{\sigma_3}{6\beta\lambda + \mu_3 + \alpha} \right] \]

\[ D_3 = 1 - 5\beta\lambda \left[ \frac{\sigma_1}{5\beta\lambda + \mu_1 + 2\alpha} + \frac{\sigma_2}{5\beta\lambda + \mu_2 + 2\alpha} + \frac{\sigma_3}{5\beta\lambda + \mu_3 + 2\alpha} \right] \]

\[ D_4 = 1 - 4\beta\lambda \left[ \frac{\sigma_1}{4\beta\lambda + \mu_1 + 3\alpha} + \frac{\sigma_2}{4\beta\lambda + \mu_2 + 3\alpha} + \frac{\sigma_3}{4\beta\lambda + \mu_3 + 3\alpha} \right] \]

\[ Y = 7\beta\lambda(a_1 + a_2 + a_3)(c_1 + c_2 + c_3 - 1) + 6\beta\lambda(b_1 + b_2 + b_3)(c_1 + c_2 - 1) + 5\beta\lambda(d_1 + d_2 + d_3)(c_1 - 1) + 8\lambda(c_1 + c_2 + c_3 - 1) \]

Now, from (13),

\[ P_0 = 1/(1 + a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + d_1 + d_2 + d_3 + e_1 + e_2 + e_3 + f_1 + f_2 + f_3) \]

Therefore, the expected number of units in the system and in the queue are, respectively,
\[ L = \sum_{m=1}^{5} \sum_{i=1}^{3} m P_{m,i} \]
\[ = \{a_1 + a_2 + a_3 + 2(b_1 + b_2 + b_3) + 3(d_1 + d_2 + d_3) + 4(e_1 + e_2 + e_3) + 5(f_1 + f_2 + f_3)\}P_0 \]
\[ L_q = \sum_{m=2}^{5} \sum_{i=1}^{3} (m-1) P_{m,i} = L + P_0 - 1 \]
\[ = \{b_1 + b_2 + b_3 + 2(d_1 + d_2 + d_3) + 3(e_1 + e_2 + e_3) + 4(f_1 + f_2 + f_3)\}P_0. \]

The machine availability (rate of production per machine) is
\[ M. A. = 1 - \frac{L}{5}. \]
The operative efficiency (utilization) is
\[ O. E. = 1 - P_0. \]
Moreover, if we put \( \sigma_1 = 0.3, \sigma_2 = 0.2, \sigma_3 = 0.5, \mu_1 = 4, \mu_2 = 3, \mu_3 = 2, \lambda = 6, \beta = 0.3, \alpha = 0.6, c_1 = 0.2, c_2 = 0.1, c_3 = 0.3, c_4 = 0.15 \) and \( c_5 = 0.25 \), we get:

Special Cases
Some modeling systems can be obtained as special cases of this system:
(i) Let \( \sigma_r = \delta_{rs} \) and \( \mu_r = \mu \) where \( \delta_{rs} \) is the Kronecker delta function, then we get the Markovian machine interference system: \( M^X/M/1/k/N \) with bulk arrivals, balking and reneging, and the results are:
\[ P_1 = \frac{N\lambda}{\mu} P_0, \]
\[ P_m = \frac{\beta\lambda}{\mu + (m-1)\alpha} \sum_{j=1}^{m-1} (N-m+j)c_j P_{m-j} - \frac{R(m)}{\mu + (m-1)\alpha}, m = 2(1)k-1, \]
\[ P_k = \frac{\lambda}{\mu + (k-1)\alpha} \left\{ \beta\lambda \sum_{j=1}^{k-1} (N-k+j)c_j P_{k-j} + N\lambda c_k P_0 + \beta\lambda \sum_{j=2}^{k} \sum_{n=k-j+1}^{m-1} (N-n)c_j P_n \right\} \]
where
\[ R(m) = -\beta\lambda \sum_{j=1}^{m-1} (N-j)P_j + \beta\lambda \sum_{n=1}^{m-1} \sum_{j=1}^{m-n} (N-j)c_n P_j + N\lambda P_0 \sum_{j=1}^{m-1} c_j - N\lambda P_0. \]

(ii) If we put \( c_j = \delta_{j1} \), we get the system: \( M/H^r_{\infty}/1/k/N \) with balking and reneging which studied by Shawky [13].

References