Application Of Navier-Stokes Equations Via A Model For Water Flow In Green Plant

[1] Lawal O.W, [2] Adeosun T.A, [3] Olayiwola M.A, and [4] Falade A.O
[1,2,3,4] Department of Mathematics

[1] Tai Solarin University of Education, Ijagun, Nigeria.
[2,4] Yaba College of Technology, Yaba, Lagos,

[3] Osun State University, Oshogbo, Nigeria.
adebaba2001@yahoo.com

Abstract: A dynamic mechanistic mathematical model of water flow through the xylem of growing plant is developed. We describe current theories about the physiology of xylem that necessitated the development of internal processes to supply all part of the plant with water. The model via Navier-Stokes equations is known to be a good tool for interpreting the phenomena of water transport in vascular tissues from roots hairs to the leaves. We derive the Hagen-Poiseuille formula for circular cross sectional xylem and determine the amount of flow in an annular cross sectional xylem. Some results of the model show that the increase in boundary of the annular cross sectional xylem compared with the circular-sectional xylem makes the coefficient in the velocity equation different. Hence a normalized annular xylem is presented where outer radius of xylem is fixed and the inner radius varied. [Journal of American Science 2010;6(10):795-798]. (ISSN: 1545-1003).

Keywords: dynamic mechanistic mathematical model; xylem; plant

1. Introduction

A Navier-Stokes equations (Lapidus and Pinder, 1999) is a model example of Newton`s law of motion. It is a good tool for interpreting some interesting phenomena appearing in engineering flows. Nevertheless, the coverage is limited to physiology of xylem (Thornley, 1979), since non-linear partial differential equations are difficult to solve analytically with few exceptions. However, due to the widespread use of computers, obtaining any numerical solutions is feasible. The simplest one will be finite element method which includes several versions. Another reason for limiting the coverage lies in difficulty in finding easy and interesting examples beyond Hagen-Poiseuille's law (HP) for quasi-static pressure gradient.

Let us compare the xylem system of plants with the more familiar human vascular system. In contrast to the human circulatory system, the xylem system of plants is responsible for the transport of water from the roots throughout the plant. Water transport in vascular tissue where xylem carries water and mineral ions from roots hairs to the leaves (Kosh, Stephen and Gregory, 2004). Unlike the blood vessels of human physiology, the conduits of plants are formed of individual plant cells placed adjacent to one another. During cell differentiation the common walls of two adjacent cells develop holes, which permit fluid to pass between them. It has been proven that water moves across each cell by osmosis (Landsberg ans Fowkes, 1979), (Fred, 1981) provided the mechanism that forces water up the plant in various ways – when xylem is dead at its mature stage then water travels up the plant, not due to any pumping mechanism by live cell. Pericycle provides the force for lifting the water up the xylem is a process called transpiration stream and this moves against the force of gravity because of root pressure and capillary action.

(Upadhyaya, 2004, 2002 and 1998) claimed that roots are not needed; the solution was drawn up the trunk, killing nearby tissues as it went. Leaves are needed for upward movement of water. Hence this transport is not powered by energy spent by the tracheary element contained in the xylem while the phloem contains sieve element that remain metabolically active, (Northington and Goodin, 1984).

Blood vessels are often modeled as elastic tubes since their deformation may be significant due to the pulsatile nature of the flow. In plants, however, the flow is quasi-steady and the vascular cells have stiff cell walls, making a rigid-tube model appropriate. Reynolds numbers for flow in the human aorta and in the xylem of a plant are respectively about 2000 and 0.02. This means that there exists slow viscous flow. Kenyon, (1960,), Slatyer, (2000) whereby the inertia terms are neglected in the Navier-Stokes equations. In this paper, the Hagen – Poiseuille formula for circular cross sectional xylem using the Navier-Stokes and continuity equation was derived with the determination of amount of flow in an annular cross sectional xylem.

Model Formulation For Water Flow In Xylem

For viscous incompressible fluid, a combination of the Navier-Stokes equation and the Continuity equation are given as

$$\rho \frac{dv}{dt} = \rho_x - \nabla_1 + \mu \nabla^2 v + \frac{\mu}{3} \nabla(\nabla \cdot V) \quad (1)$$

and

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \quad (2)$$

For incompressible fluids, the last term of equation (1) reduces to

$$(\mu/3)\nabla(\nabla \cdot V) = 0$$

$$and \;\; \rho_x - \nabla_\rho = \nabla_\rho$$

The derivation of a formula for the flow of water in the xylem of a plant, requires that we write out the Navier-Stokes equation and the continuity equation in cylindrical coordinators. This shows that the pressure difference changes only in the z - direction while the velocity in the z - direction remains constant. Therefore reduce the Navier-Stokes equation to a simplified equation in cylindrical coordinates (r, θ, z) are given as:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right) = \frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) \quad (3)$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_0}{\partial r} - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) \quad (4)$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) \quad (5)$$

where $p \; and \; v$ respectively are the pressure and velocity and ρ is the mass density and μ the coefficient of viscosity with the assumption that there is no rotational velocity, hence we have

$$v_\theta = 0, \qquad v_r = 0 \quad (6)$$

and that the flow is steady state, and even flow

$$\frac{\partial v_r}{\partial t} = 0, \qquad \frac{\partial v_\theta}{\partial t} = 0, \qquad \frac{\partial v_z}{\partial t} = 0 \quad (7)$$

Apply Equ (7) to Equ (3) and (4) gives

$$\frac{\partial p}{\partial r} = 0$$
$$\frac{\partial p}{\partial \theta} = 0 \qquad (8)$$

Then Equ (5) becomes

$$\rho\left(v_z \frac{\partial v_z}{\partial z}\right) = \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z}\right) \qquad (9)$$

Equation (2) in cylindrical coordinates gives

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad (10)$$

Applying (6) to Equ (10) gives

$$\frac{\partial v_z}{\partial z} = 0$$

Thus Equ (9) simplified to

$$\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}\right) = 0 \qquad (11)$$

we assume that the velocity is the z - direction with respect to θ is constant.

Therefore $\frac{\partial v_z}{\partial \theta} = 0,$ this implies that $\frac{\partial^2 v_z}{\partial \theta^2} = 0$

Equation (11) then becomes

$$\mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) = -\frac{\partial p}{\partial z} \qquad (12)$$

Note that $-\frac{\partial p}{\partial z}$ depend on z since $\frac{\partial p}{\partial r} = 0 \;\; and \;\; \frac{\partial p}{\partial \theta} = 0$; but the left hand side of Equ (12) does not depend on z so the two sides are constant.

Therefore (12) can be rewrite as

$$\mu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr}\right) = -\frac{dp}{dz} \qquad (13)$$

On integration, the result appears to be

$$V = -\frac{r^2}{4\mu}\frac{dp}{dz} + a\log r + b \qquad (14)$$

The constant a must be equal to zero since the velocity must remain finite at the centre of the xylem. The constant b is determined from the requirement that $v = 0$ for $r = R,\ where\ R$ is the radius of the xylem.

Therefore $$v = \frac{1}{4\mu}\frac{dp}{dz}(R^2 - r^2) \qquad (15)$$

and the expression for volume rate of flow is given

by $$Q = \int_0^R 2\pi r V dr \qquad (16)$$

which gives

$$Q = \frac{\pi}{8\mu}\frac{dp}{dz}R^4 \qquad (17)$$

Hence the Hagen Poiseuille formular for a circular pipe is derived.
Now to determine the amount of flow in an annular cross sectional xylem, the computation is the same as for the circular cross-sectional xylem.

Let $v = 0 \quad when \quad r = R_1 \quad and \quad r = R_2$

Where $R_1 \; and \; R_2$ are inner radius and outer radius respectively.

Equation (15) can be rewritten as

$$V = \frac{1}{4\mu}\frac{dp}{dz}\left(R_2^2 - r^2 + \frac{R_2^2 - R_1^2}{\log R_2/R_1}\log\frac{r}{R_2}\right) \tag{18}$$

And for volume rates of flow, we have

$$Q = \frac{\pi}{8\mu}\frac{dp}{dz}\left[R_2^4 - R_1^4 - \frac{\left(R_2^2 - R_1^2\right)^2}{\log\left(R_2/R_1\right)}\right] \tag{19}$$

Equation (19) can be rewritten as

$$f = 1 - m^{-4} - \frac{\left(1 - m^{-2}\right)^2}{\log m} \tag{20}$$

where $f = \frac{Q_a}{Q_c} \quad and \quad m = \frac{R_2}{R_1}$,

where Q_a denotes the flow in the annular region of xylem.

$R_1 \le r \le R_2 \; and \; Q_c$ is the flow in a cylindrical region of radius R_2.

Results

We noted that equations (18 and 20) hold for laminar incompressible flow. The Reynolds number should be smaller than 0.02 otherwise the transition and/or turbulence will be set up.

The flow in both circular cross sectional xylem and annular cross-sectional xylem is reduced to equation (20). A resulting graphical normalized annular xylem is shown in Figure 1. The result shows a normalized annular xylem where R_2 is fixed and R_1 ranges from $0 \; to \; R_2$. Hence the flow rate of water in xylem is affected by chemical concentration and hydrostatic pressure gradient.

Conclusion

In this paper, we have introduced a simple model for flow rate on the basis of the simplified version of Navier Stokes equations. In the physiology of xylem in plants, it is observed that this model will contribute immensely towards the effective flow of water in green plants. This model will serve as a second fiddle in investigating other similar problems of cavitation column of water in the xylem that may interrupt its flow. One possible problem will be in the water flow in curved phloem of green plants.

Nomeclature

$p =$ Pressure gradient
$\rho =$ Mass density
$v =$ Velocity
$\mu =$ Coefficient of viscosity
$Q =$ Volume flow rate
$R = r =$ Radius of cross-sectional xylem
$r, \theta, z =$ Cylindrical coordinates

Acknowledgements
We are also grateful to Professor R.O. Ayeni for his contribution.

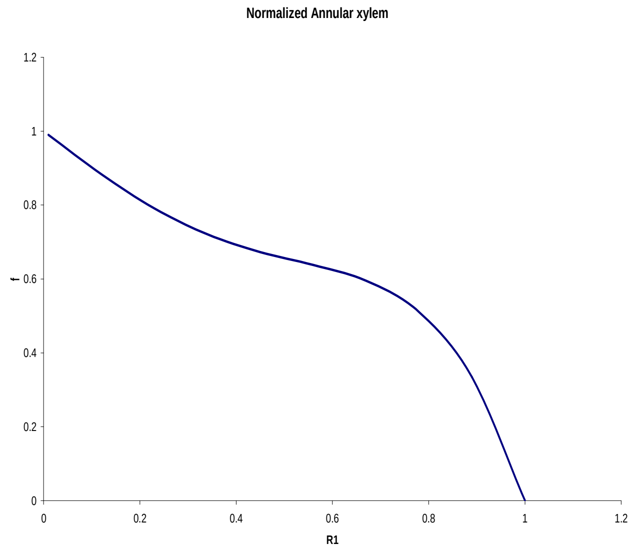

Fig 1: Normalized annular xylem

References

1. Lapidus, L and G. F Pinder. Numerical Solution of Partial Different Equations in Science and Engineering, Wiley Interscience, New York.(1999): 20-100
2. Thornley, J.H.M. Mathematical Models in Plant Physiology. New York. (1979):56-58
3. Koch , G.W , Stephen, C..S, Gregory, M.J . The Limit to Tree Height: Academic press.(2004)
4. Landsberg, J.J and Fowkes, N.D . Water Movement through Plant Roots Annals of Botany. (1978); 42: 493-508
5. Fred, J. Molz. Models of Water Transport in the Soil plant System: Water Resource Research.(1981); 17: 1245-1260.
6. Upadhyaya, S.K, Rand, R.H and Cooke, J.R. The role of stomata oscillations on plant productivity and water use efficiency. Stuttgart, Teubner. (2004) : 24-30
7. Upadhyaya, S.K, Rand, R.H and. Cooke, J.R. A mathematical model of the effects of CO_2 on Stomata dynamics, New York. (2002): 49-51.
8. Upadhyaya, S.K, Rand, H.R and Cooke, J.R. Stomata dynamics. New York. (1998): 44-45
9. Northington, D.K and Goodin, J.R. The Botanical World; Mirror/Mosby College Publishing, St. Louis. (1984)
10. Kenyon, R.A. Principle of Fluid Mechanics; Roland Press Company, New York. (1960)
11. Jahnke,E, Emde, F and Losch, F. Table of Higher Functions: 7th ed., Stuttgart, Teubner.(1996); 14-34
12. Slatyer, R.O and Taylor, S.A: Terminology in Plant and Soil-Water Relation; Nature.(2000); 187: 922-924.
11. Webster, I.A: The use of the effectiveness factor concept in CO_2 diffusion in the leaf interior. Stuttgart, Teubner. (1999);10-13

.
7/7/2010