Approximate Optimal Control for a Class of Nonlinear Volterra Integral Equations

Akbar H. Borzabadi, Akram Abbasi and Omid S. Fard

Department of Applied Mathematics, Damghan University, Damghan, Iran
borzabadi@dubs.ac.ir

Abstract: In this study an iterative approach to extract approximate solutions of optimal control problems which are governed by a class of nonlinear Volterra integral equations is presented. The structure of approach is based on the parametrization of the control and state functions. Considering some conditions on the problem, the convergence of the given approach is studied. Numerical examples illustrate the efficiency of the given approach.
[Akbar H. Borzabadi, Akram Abbasi and Omid S. Fard. Approximate Optimal Control for a Class of Nonlinear Volterra Integral Equations. Journal of American Science 2010;6(11):1017-1021]. (ISSN: 1545-1003). (http://www.americanscience.org).

Keywords: Optimal control; Volterra integral equations; iterative schemes; approximation

1. Introduction

The classical theory of optimal control was developed in the last years as a powerful tool to create optimal solutions for real processes in many aspects of science and technology. Complexity of applying analytical methods for obtaining fast and near optimal solutions is the reason for creating numerical approaches. An overview of numerical methods for solving optimal control problems described by ODE and integral equations can be found in (Schmidt, 2006). However these methods are not much developed for optimal control of nonlinear integral equations. Belbas (1999; 2007; 2008) has introduced and elaborated some interesting iterative schemes with their convergence for optimal control of Volterra integral equations considering some conditions on the kernel of integral equation. Also some methods based on approximating the kernel of integral equation which gives rise to a system of ordinary differential equations for approximating the Volterra integral equation can be seen in (Lukas and Teo, 1991; Wu *et al*., 2007).

Of course it seems that the lack of general methods for solving Volterra integral equations makes serious difficulties in using of these schemes.

The idea of combination of some numerical methods for solving optimal control problems and Volterra integral equations may lead to present executable numerical approaches for obtaining near optimal solutions of optimal control problems governed by Volterra integral equations. This study intends to actualize this idea by combining the method of parameterization, (Mehne and Borzabadi, 2006; Teo *et al*., 1999a; 1999b) and the method of power series, (Maleknejad *et al*., 2007a; 2007b; Tahmasbi and Fard, 2008), which are successful methods for solving some classes of optimal control problems and Volterra integral equations, respectively, for providing a numerical scheme to find approximate optimal control of systems governed by some classes of nonlinear Volterra integral equations which can be described by the following minimization problem:

$$\text{Minimize } J(x,\ u) = \int_0^T \zeta(t, x(t), u(t))dt \tag{1}$$

Subject to:

$$x(t) = y(t) + \int_0^t k(s, t.x(s))ds, \text{ a.e. on } [0,T] \tag{2}$$

where, $x(\cdot), u(\cdot) \in C^{\infty}([0,T])$, $\zeta \in C([0,T]\times \mathrm{P} \times \mathrm{P})$ and $\kappa \in C([0,T]\times \mathrm{P}\times \mathrm{P}\times \mathrm{P})$.

After this and without loss of generality we suppose $T = 1$. Analytical discussions about existence and uniqueness of the optimal control of systems governed by nonlinear Volterra integral equations can be found in (Angell, 1976).

2. The control and state parameterization

Let Q be the subset of product space $C^{\infty}([0,1])\times C^{\infty}([0,1])$ contains all pairs $(x(\cdot),u(\cdot))$ that satisfy in the integral Eq. 2. Also let $Q_{m,n}$ be the subset of Q consisting of all pairs $(x_m(\cdot),u_n(\cdot))$ where $u_n(\cdot)$ is a parameterized control function as the following polynomial:

$$u_n(t) = \sum_{i=0}^{n} a_i t^i \tag{3}$$

And $x_m(\cdot)$ is extracted solution of the integral equation:

$$x(t) = y(t) + \int_0^t k(s, t, x(s), u_n(s))ds \tag{4}$$

and it is considered as a polynomial of degree at most m:

$$x_m(t)=\sum_{j=0}^{m} e_j(\alpha_0,\alpha_1,...,\alpha_n)t^j \qquad (5)$$

such that $e_j : P^n \to P$, $j=1,2,\cdots,m$, are continuous functions. Now we consider the minimizing of J on $Q_{m,n}$ with $\{a_k\}_{k=0}^{n}$ as unknowns. This is obviously an optimization problem in n dimensional space:

$$\{(\alpha_0,\alpha_1,...,\alpha_n)\in \mathbb{R}^{n+1} : \alpha_0 = u_n(0)=u_0, \sum_{k=0}^{n}\alpha_k = u_n(1)=u_1\}$$

and $J(x_m,u_n)$ may be considered as a function $J(a_0,a_1,\cdots,a_n)$.

Suppose $(x_m^*(\cdot),u_n^*(\cdot))$ be the solution of minimizing J on $Q_{m,n}$, $m=1,2,\cdots$, $n=1,2,\cdots$ then polynomial form of $u_n^*(\cdot)$, $n=1,2,...$ in (3) and considering the special form of integral equation kernel allow us applying a method based on power series for extracting polynomial solution of (4), (Tahmasbi and Fard, 2008), where applying this method give rise to obtain a sequence of state functions $\{x_m^*(\cdots)\}_{m=1}$ as Taylor series, see Theorem 1 in (Tahmasbi and Fard, 2008) and finally to achieve a minimizing sequence $\{(x_m^*(\cdot)\},u_n^*(\cdot))\}_{m,n}$.

Lemma 1: If $\alpha_{m,n}=\inf_{Q_{m,n}} J$ for $m,n=1,2,\cdots$, then $\{\alpha_{m,n}\}_{m,n=1}^{\infty}$ is a convergent sequence.

Proof: By definition $Q_{m,n}$ we have:

$$Q_{1,1}\subset Q_{1,2}\subset Q_{2,2}\subset\cdots\subset Q_{m,n}\subset Q_{m+1,n}\subset\cdots\subset Q,$$

and therefore:

$$\alpha_{1,1}\ge\alpha_{1,2}\ge\alpha_{2,2}\ge\cdots\ge\alpha_{m,n}\ge\alpha_{m+1,n}\ge\cdots\ge\alpha,$$

Now it can be concluded that $\{\alpha_{m,n}\}$ is convergent because it is a no decreasing and bounded from below sequence.

Theorem 1: If $\lim_{m,n\to\infty}\alpha_{m,n}=\alpha$ then $\alpha=\inf_Q J$.

Proof: By Lemma 1, let $\{\alpha_{m,n}\}$ is convergent to namely $\hat{\alpha}\ge\alpha$. By contradiction if $\hat{\alpha}\ge\alpha$, then $\in=\frac{\hat{\alpha}\ge\alpha}{2}>0$. By the properties of infimum, (Rudin, 1976), there exists $(x(\cdot),u(\cdot))$, such that:

$$J(x(\cdot),u(\cdot))<\alpha+\in=\frac{\hat{\alpha}}{2}+\frac{\alpha}{2} \qquad (6)$$

From the continuity of J, there is a $\delta>0$ where:

$$|J(v(\cdot),w(\cdot))-J(x(\cdot),u(\cdot))|<\varepsilon, \qquad (7)$$

Whenever:

$$\|(v(\cdot),w(\cdot))-(x(\cdot),u(\cdot))\|_\infty<\delta. \qquad (8)$$

Here $\|\|_\infty$ is a norm on the vector space $C^\infty([0,1])\times C^\infty([0,1])$ which can be defined as follows:

$$\|(v(\cdot),w(\cdot))\|_\infty=\|v(\cdot)\|_\infty+\|w(\cdot)\|_\infty,$$

and one can easily check the properties of the norm for it. On the other hand side, the set of all polynomial pairs are dense in $C^\infty([0,1])\times C^\infty([0,1])$, so there is a pair of polynomials pm(t) of degree at most m and $q_n(t)$ of degree at most n such that:

$$\|(p_m(\times),\ q_n(\times))-(x(\times),\ u(\times))\|_\infty<\frac{\delta}{3}. \qquad (9)$$

Whereas the pair $(p_m(\cdot),q_n(\cdot))$ does not satisfy:

$$(p_m(0),q_n(0))=(x_0,u_0),(p_m(1),q_n(1))=(x_1,u_1),$$

We have to define another polynomials:

$$v_m(t)=p_m(t)+(x_0-p_m(0))(1-t)+(x_1-p_m(1))t,$$
$$w_n(t)=q_n(t)+(u_0-q_n(0))(1-t)+(u_1-q_n(1))t,$$

that satisfy $(v_m(0),w_n(0))=(x_0,u_0)$ and $(v_m(1),w_n(1))=(x_1,u_1)$, so $(v_m,w_n)\in Q_{m,n}$.

From (9) for t = 0,1 we have:

$$\|(p_m(0),\ q_n(0))-(x_0,\ u_0)\|_\infty<\frac{\delta}{3},$$
$$\|(p_m(1),\ q_n(1))-(x_1,\ u_1)\|_\infty<\frac{\delta}{3}$$

Now for $t\in[0,1]$ by definition $v_m(\cdot)$ and $w_n(\cdot)$ we have:

$$\|(v_m(\cdot),w_n(\cdot))-(x(\cdot),u(\cdot))\|_\infty\le\|(p_m(t),q_n(t))-(x(t),u(t))\|_\infty$$
$$\|(p_m(0),q_n(0))-(x_0,u_0)\|_\infty(1-t)+\|(p_m(1),q_n(1))-(x_1,u_1)\|_\infty t$$

$$<\frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3}=\delta.$$

Therefore:

$$\|(v_m(\cdot),w_n(\cdot))-(x(\cdot),u(\cdot))\|_\infty<\delta,$$

and (7-8) imply that:

$$|J(v_m(\cdot),w_n(\cdot))-J(x(\cdot),u(\cdot))\|_\infty<\varepsilon,$$

and so from (6):

$$J(v_m(\cdot),w_n(\cdot))\ <\frac{\hat{\alpha}}{2}-\frac{\alpha}{2}+J(x(\cdot),u(\cdot))<\hat{\alpha}$$

a contradiction concludes with $(v_m(\cdot), w_n(\cdot)) \in Q_{m,n}$, so $\hat{\alpha} = \alpha$.

Now we summarize the above results in a numerical algorithm for obtaining approximate optimal control of minimizing (1) subject to (2).

Algorithm 1: Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ for accuracy of the solution.

Step 1: Let $m,n,k = 1$, $u_1(t) = a_0 + a_1 t$, $x_1(t) = e_0 + e_1 t$ and $\alpha_1 = J(x_1(\cdot), u_1(\cdot))$, where $e_0 = e_0(a_0, a_1)$ and $e_1 = e_1(a_0, a_1)$.

Step 2: Let $m \to m+1$ and $k \to k+1$ and find $\alpha_k = \inf_{Q_{m,n}} J$.

Step 3: If $|\alpha_k - \alpha_{k-1}| < \varepsilon_1$ then go to Step 4, otherwise go to Step 2

Step 4: Let $n \to n+1$ and $k \to k+1$ and go to Step 2

Step 5: If $|\alpha_k - \alpha_{k-1}| < \varepsilon_1$ then stop, otherwise go to Step 4

3. Numerical results

In this section some examples show the interesting results of the proposed iterative approach.

Example 1: In the first example we consider the optimal control of minimizing:

$$J = \int_0^1 \left(\left(x(t) - \cos(t) \right)^2 + \left(u(t) - t \right)^2 \right) dt \tag{10}$$

subject to the following nonlinear Volterra integral equation:

$$x(t) = y(t) + \int_0^t u(s)^2 \left(x(s) + ts \right) ds \tag{11}$$

Where:

$$y(t) = \cos(1-2t) + \sin\left(t\left(2-t^2\right) \right) - \frac{1}{4} t^5$$

The exact optimal solution of problem (10-11) are $x^*(t) = \cos(t)$ and $u^*(t) = t$ with optimal criteria $J^* = J(x^*(t), u^*(t)) = 0$. We apply Algorithm 1 on this problem. Let $m,n = 1$, so we have:

$$u(t) = a_0 + a_1(t),\ x(t) = e_0 + e_1 t$$
$$\Rightarrow x(t) = 1 + e_1 t,\ (e_0 = y(0)).$$

Substituting $x(t)$ and $u(t)$ in (11) concludes that $e_1 = \alpha_0^2$. To optimize J on (a_0, a_1) gives rise to $a_0 = 0, a_1 = 1$ and so $e_1 = 0$ and $\alpha_{1,1} = 0.0444$. Now let $m = 2$ and $n = 1$. We have:

$$u(t) = \alpha_0 + \alpha_1 t,\ x(t) = 1 + \alpha_0^2 t + e_2 t^2$$

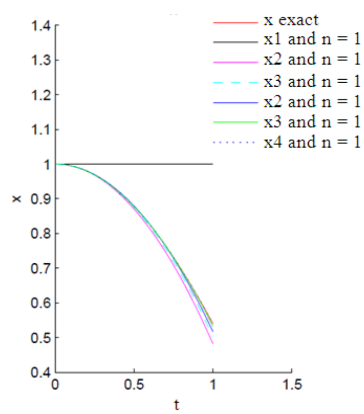
and again substituting $x(t)$ and $u(t)$ in (11) conclude $e_2 = \frac{1}{2}\alpha_0^4 + \alpha_0\alpha_1 - \frac{1}{2}$ and optimizing J eventuate the following results:

$$a_0 = -0.0187, a_1 = 0.9612,$$
$$e_1 = 3.4969 \times 10^{-4}, e_2 = -0.5180$$

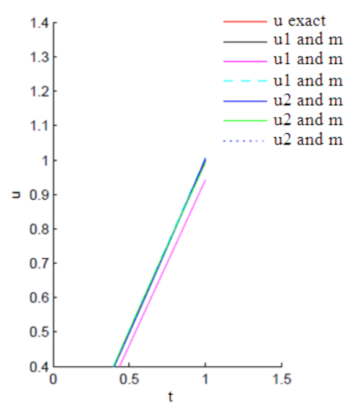
and $\alpha_{2,1} = 0.0107$. In Table 1 the successive applying of Algorithm 1 for some values of m and n is shown. Also the state and control functions that are obtained in process of using Algorithm 1 for problem (10-11) are shown in Fig. 1.

Table 1: The results of applying proposed algorithm in Example 1

n	m	$\alpha_{m,n}$
1	1	0.0444
1	2	0.0107
1	3	1.5226×10^{-4}
2	2	7.8920×10^{-5}
2	3	2.5166×10^{-5}
2	4	1.0922×10^{-7}



(a)



(b)

Fig. 1: The state and control functions in example 1

Example 2: In this example the optimal control problem of minimizing

$$\text{Minimize } J = \int_0^1 ((x(t) - \sin(t))^2 (u(t)\text{-}t)^2)dt \qquad (12)$$

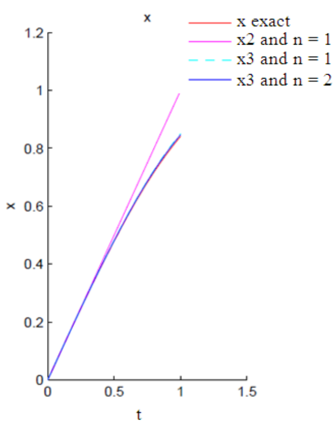
governed by the nonlinear Volterra integral equation:

$$x(t) = y(t) + \int_0^t u(s)(x(s) + t)ds \qquad (13)$$

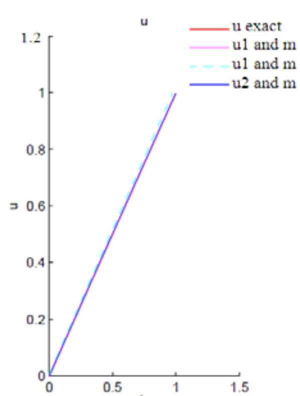
where, $y(t) = t\cos(t)\text{-}\frac{1}{2}t^3$ is considered. The results of successive applying Algorithm 1 on this problem, as previous example, is shown in Table 2. Also the state and control functions obtained during the application of Algorithm 1 on problem (12-13) are shown in Fig. 2.

Table 2: The results of applying proposed algorithm in Example 2

n	m	$\alpha_{m,n}$
1	2	3.5575×10^{-11}
1	3	1.4975×10^{-12}
2	3	3.0905×10^{-14}



(a)



(b)

Fig. 2: The state and control functions in example 2

Example 3: It seems to obtain suitable approximate solution for problems that have the exact solution as exponential function is difficult. In this example we consider the optimal control problem of minimizing nonlinear functional:

$$\text{Minimize } J = \int_0^1 ((x(t)\text{-}e^t)^2 (u(t)\text{-}e^t)^2)dt \qquad (14)$$

on the Volterra integral equation:

$$x(t) = y(t) + \int_0^t u(s)(x(s) + t)ds \qquad (15)$$

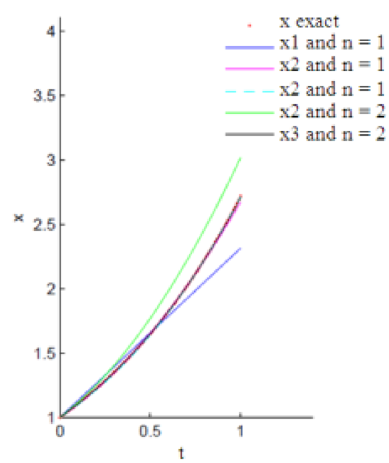
Where:

$$y(t) = e^t(1\text{-}t\text{-}\frac{1}{2}e^t) + t + \frac{1}{2}$$

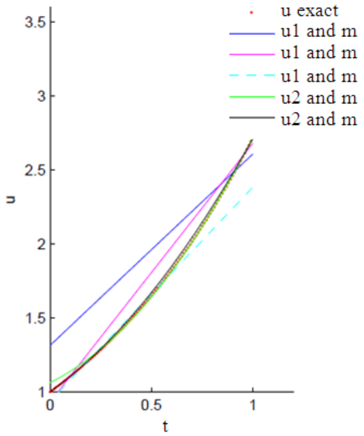
The exact optimal control and state functions are $u^*(t) = e^t$ and $x^*(t) = e^t$ respectively and optimal criteria is $J^* = J(x^*(t), u^*(t)) = 0$. Table 3 shows interesting results by applying proposed approach on the problem (14-15). In Fig. 3, one can see the state and control functions that are obtained during the process of applying Algorithm 1 on problem (14-15).

Table 3: The results of applying proposed algorithm in example 3

n	m	$\alpha_{m,n}$
1	1	2.1070×10^{-4}
1	2	2.3385×10^{-7}
1	3	1.9795×10^{-8}
2	2	1.4383×10^{-8}
2	3	4.2201×10^{-10}



(a)



(b)

Fig. 3: The state and control functions in example 3

4. Conclusion

In this study, we have proposed a numerical scheme for finding approximate solution of optimal control problems governed by a class of nonlinear Volterra integral equations. Our limitation in the application of this method depends on the type of integral equations that we face because of the power series method only for solving certain categories of integral equations may be applied. Although the presented numerical examples show the efficiency of the method for solving a wide range of problems.

Corresponding Author:
Dr. Akbar H. Borzabadi
Department of Applied Mathematics
Damghan University
Damghan, Iran
E-mail: borzabadi@dubs.ac.ir

References

1. Schmidt WH. Numerical methods for optimal control problems with ODE or integral equations. Lecture Notes Comput. Sci. 2006; 3743: 255-262.
2. Lukas MA, Teo KL. A computational method for a general class of optimal control problems involving integrodifferential equations. Optimal Control Applied Meth. 1991; 12: 141-162.
3. Wu CZ, Teo KL, Zhao Y, Yan WY. An optimal control problem involving impulsive integrodifferential systems. Optimizat. Meth. Software 2007; 22: 531-549.
4. Mehne HH, Borzabadi AH. A numerical method for solving optimal control problems using state parametrization. J. Numerical Algorithms 2006; 42: 165-169.
5. Teo KL, Jennings LS, Lee HWJ, Rehbock V. Control parametrization enhancing technique for constrained optimal control problems. J. Austral. Math. Soc. B. 1999a; 40: 314-335.
6. Teo KL, Goh CJ, Wong KH. A unified computational approach to optimal control problems. Longman Scientific Technical, Essex, 1999b.
7. Maleknejad K, Kajani MT, Mahmoudi Y. Numerical solution of Fredholm and Volterra integral equations of the second kind by using Legendre wavelets. Kybernetes 2007a; 32: 1530-1539.
8. Maleknejad K, Sohrabi S, Rostami Y. Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials. Applied Math. Comput. 2007b; 188: 123-128.
9. Tahmasbi A, Fard OS. Numerical solution of nonlinear volterra integral equations of the second kind by power series. J. Inform. Comput. Sci. 2008; 3: 57-61.
10. Angell TS. On the optimal control of systems governed by nonlinear Volterra equations. J. Optimizat. Theory Appl. 1976; 19: 29-45.
11. Rudin W. Principles of Mathematical Analysis. 3rd Edn. McGraw-Hill, 1976.
12. Belbas SA. Iterative schemes for optimal control of Volterra integral equations. Nonlinear Analysis: Theory. Meth. Appl. 1999; 37: 57-79.
13. Belbas SA. A new method for optimal control of Volterra integral equations. Applied Math. Comput. 2007; 189: 1902-1915.
14. Belbas SA. A reduction method for optimal control of Volterra integral equations. Applied Math. Comput. 2008; 197: 880-890.

8/1/2010