Cubic Nonpolynomial Spline Approach to the Solution of a Second Order Two-Point Boundary Value Problem

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Abstract: Third and fourth order convergent methods based on cubic nonpolynomial spline function at midknotes are presented for the numerical solution of a second order two-point boundary value problem with Neumann conditions. Using this spline function a few consistency relations are derived for computing approximations to the solution of the problem. Convergence analysis of these methods is discussed two numerical examples are given to illustrate practical usefulness of the new methods. [Journal of American Science. 2010;6(12):297-302]. (ISSN: 1545-1003).

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1. Introduction:

In approximation theory spline functions occupy an important position having a number of applications, especially in the numerical solution of boundary-value problems. We shall consider a numerical solution of the following linear second order two-point boundary value problem, see [5].

$$y^{(2)} + f(x)y = g(x), \quad x \in [a,b] \qquad (1.1)$$

Subject to Neumann boundary conditions:

$$y^{(1)}(a) - A_1 = y^{(1)}(b) - A_2 = 0 \qquad (1.2)$$

Where A_i , $i = 1, 2$ are finite real constants. The functions $f(x)$ and $g(x)$ are continuous on the interval $[a,b]$. The analytical solution of (1.1) subjected to (1.2) cannot be obtained for arbitrary choices of $f(x)$ and $g(x)$.

The numerical analysis literature contains little on the solution of second order two-point boundary value problem (1.1) subjected to Neumann boundary conditions (1.2).while The linear second order two-point boundary value problem (1.1) subjected to Dirichlet boundary conditions solved by different types of spline functions, see [1, 7, 8, 9].

Ramadan et al. [5] solved the problem (1.1) subjected to (1.2) using quadratic polynomial spline, cubic polynomial spline and quadratic nonpolynomial spline at midknotes.

In this paper, we develop cubic nonpolynomial spline at midkontes to get smooth approximations for the solution of the problem (1.1) subjected to Neumann boundary conditions (1.2).

2. Derivation of the method:

We introduce a finite set of grid points x_i by dividing the interval $[a, b]$ into n equal parts.

$x_i = a + ih, \; i = 0, 1, \ldots\ldots, n$

$$x_0 = a \,,\; x_n = b \;\; and \;\; h = \frac{b-a}{n} \qquad (2.1)$$

Let $y(x)$ be the exact solution of the system (1.1) and (1.2) and S_i be an approximation to $y_i = y(x_i)$ obtained by the spline function $Q_i(x)$ passing through the points (x_i, s_i) and (x_{i+1}, s_{i+1}).

Each nonpolynomial spline segment $Q_i(x)$ has the form.

$$Q_i(x) = a_i \sin k(x - x_i) + b_i \cos k(x - x_i) + c_i (x - x_i) + d_i \,, i = 0,1,\ldots.n-1 \qquad (2.2)$$

Where a_i , b_i , c_i and d_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method and equation (2.2) reduces to cubic polynomial spline function in $[a,b]$when $k \to 0$, Choosing the spline function in this form will enable us to generalize other existing methods by arbitrary choices of the parameters α and β which will be defined later at the end of this section. Thus, our cubic nonpolynomial spline is now defined by the relations:

$$(i)\, S(x) = Q_i(x), x \in [x_i, x_{i+1}], i = 0,1,\ldots\ldots.,n-1$$

$$(ii)\, S(x) \in C^{\infty}[a,b] \qquad (2.3)$$

The four coefficients in (2.2) need to be obtained in terms of

$S_{i+\frac{1}{2}}, D_i, M_{i+\frac{1}{2}}, T_i \; and \; T_{i+1}$ Where

$$
\begin{aligned}
&(i)\; Q_i\left(x_{i+\frac{1}{2}}\right) = S_{i+\frac{1}{2}} \\
&(ii)\; Q_i^{(1)}(x_i) = D_i \qquad (2.4)\\
&(iii)\; Q_i^{(2)}\left(x_{i+\frac{1}{2}}\right) = M_{i+\frac{1}{2}} \\
&(iv)\; Q_i^{(3)}(x_i) = \frac{1}{2}[T_i + T_{i+1}]
\end{aligned}
$$

We obtain via a straightforward calculation

$$a_i = \frac{-1}{2k^3}[T_{i+1}+T_i],\; b_i = \frac{\tan\theta/2}{2k^3}[T_{i+1}+T_i] - \frac{\sec\theta/2}{k^2}M_{i+\frac{1}{2}}$$

$$C_i = D_i + \frac{1}{2k^2}[T_{i+1}+T_i],\; d_i = S_{i+\frac{1}{2}} + \frac{1}{k^2}M_{i+\frac{1}{2}} - \frac{h}{2}D_i - \frac{h}{4k^2}[T_{i+1}+T_i] \qquad (2.5)$$

Where $\theta = kh$ and $i = 0, 1, 2 \ldots n\text{-}1$

Now using the continuity conditions (*ii*) and (2.3), that is the continuity of cubic nonpolynomial spline $S(x)$ and its first and second derivates at the point (x_i, s_i), where the two cubics $Q_{i-1}(x)$ and $Q_i(x)$ join, we can have

$$Q_{i-1}^{(m)}(x_i) = Q_i^{(m)}(x_i),\; m = 0,1,2 \qquad (2.6)$$

Using Eqs. (2.2), (2.4), (2.5) and (2.6) yield the relations:

$$\frac{h}{2}[D_i + D_{i-1}] = \left(S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\right) + \frac{1}{k^2}M_{i+\frac{1}{2}}\left[1-\sec\theta/2\right] + \frac{1}{k^2}M_{i-\frac{1}{2}}\left[\cos\theta\sec\theta/2 - 1\right] + \left(\frac{\tan\theta/2}{2k^3} - \frac{h}{4k^2}\right)(T_{i-1}+2T_i+T_{i+1}) \qquad (2.7)$$

$$\frac{h}{2}[D_i + D_{i-1}] = \frac{h\sin\theta\sec\theta/2}{2k}M_{i-\frac{1}{2}} \qquad (2.8)$$

And

$$\frac{\tan\theta/2}{2k^3}(T_{i-1}+2T_i+T_{i+1}) = \frac{\sec\theta/2}{k^2}M_{i+\frac{1}{2}} - \frac{\cos\theta\sec\theta/2}{k^2}M_{i-\frac{1}{2}} \qquad (2.9)$$

From Eqs. (2.7) – (2.9) we get the following relation:

$$S_{i-\frac{3}{2}} - 2S_{i-\frac{1}{2}} + S_{i+\frac{1}{2}} = h^2\left(\alpha M_{i-\frac{3}{2}} + \beta M_{i-\frac{1}{2}} + \alpha M_{i+\frac{1}{2}}\right), i = 2,3,\ldots,n-1 \qquad (2.10)$$

Where

$$\alpha = \frac{\theta - 2\sin\theta/2}{2\theta^2\sin\theta/2} \quad And \quad \beta = \frac{2\theta\sin^2\theta/2 + 4\sin\theta/2 - \theta(1+\cos\theta)}{2\theta^2\sin\theta/2}$$

And

$$M_i = -f_iS_i + g_i \; with \; f_i = f(x_i) \, and \; g_i = g(x_i)$$

The relation (2.10) gives *(n-2)* linear algebraic equations in the *(n)* unknowns $S_{i+\frac{1}{2}}$,$i = 0,1,2, \ldots. n\text{-}1$, so we need two more equations, one at each end of the range of integration for direct computation of $S_{i+\frac{1}{2}}$. These two equations are deduced by Taylor series and the method of undetermined coefficients. These equations are

$$-hS_0^{(1)} - S_{\frac{1}{2}} + S_{\frac{3}{2}} = h^2\left(w_0M_{\frac{1}{2}} + w_1M_{\frac{3}{2}} + w_2M_{\frac{5}{2}} + w_3M_{\frac{7}{2}}\right) at\; i = 1 \qquad (2.11)$$

And

$$S_{n-\frac{3}{2}} - S_{n-\frac{1}{2}} + hS_n^{(1)} = h^2\left(w_0M_{n-\frac{1}{2}} + w_1M_{n-\frac{3}{2}} + w_2M_{n-\frac{5}{2}} + w_3M_{n-\frac{7}{2}}\right), at\; i = n \qquad (2.12)$$

Where w_i*'s* will be determined later to get the required order of accuracy.

The local truncation errors t_i, $i = 1, 2, \ldots..n$ associated with the scheme (2.10) – (2.12) can be obtained as follows, we rewrite the scheme(2.10) – (2.12) in the form

$$-hy_0^{(1)} - y_{\frac{1}{2}} + y_{\frac{3}{2}} = h^2\left(w_0y_{\frac{1}{2}}^{(2)} + w_1y_{\frac{3}{2}}^{(2)} + w_2y_{\frac{5}{2}}^{(2)} + w_3y_{\frac{7}{2}}^{(2)}\right) + t_1, at\; i = 1 \qquad (2.13)$$

$$y_{i-\frac{3}{2}} - 2y_{i-\frac{1}{2}} + y_{i+\frac{1}{2}} = h^2\left(\alpha y_{i-\frac{3}{2}}^{(2)} + \beta y_{i-\frac{1}{2}}^{(2)} + \alpha y_{i+\frac{1}{2}}^{(2)}\right) + t_i, at\; i = 2,3,\ldots,n-1 \qquad (2.14)$$

And

$$y_{n-\frac{3}{2}} - y_{n-\frac{1}{2}} + hy_n^{(1)} = h^2\left(w_0y_{n-\frac{1}{2}}^{(2)} + w_1y_{n-\frac{3}{2}}^{(2)} + w_2y_{n-\frac{5}{2}}^{(2)} + w_3y_{n-\frac{7}{2}}^{(2)}\right) + t_n, at\; i = n \qquad (2.15)$$

The terms $y_{i-\frac{1}{2}}$ and $y_{i-\frac{1}{2}}^{(2)}$ …in Eq. (2.14) are expanded around the point x_i using Taylor series and the expressions for t_i, $i = 2, \ldots n\text{-}1$ can be obtained. Also, expressions for t_i ; $i = 1, n$ are obtained by expanding Eqs. (2.13) and (2.15) around the point x_0 and x_n, respectively, using Taylor series and the expressions for $t_i\ ;\ i = 1, n$ can be obtained as

$$t_i=\begin{cases}\left[1-(w_0+w_1+w_2+w_3)\right]h^2y_i^{(2)}+\left[\frac{26}{48}-\frac{(w_0+3w_1+5w_2+7w_3)}{2}\right]h^3y_i^{(3)} \\ +\left[\frac{80}{384}-\frac{(w_0+9w_1+25w_2+49w_3)}{8}\right]h^4y_i^{(4)}+\left[\frac{242}{3840}-\frac{(w_0+27w_1+125w_2+343w_3)}{48}\right]h^5y_i^{(5)} \\ +\left[\frac{728}{46080}-\frac{(w_0+81w_1+625w_2+2401w_3)}{384}\right]h^6y_i^{(6)}+O(h^7);i=1,n \qquad (2.16) \\ \\ (1-2\alpha-\beta)h^2y_i^{(2)}+\left(\alpha+\frac{\beta}{2}-\frac{1}{2}\right)h^3y_i^{(3)}+\left(\frac{5}{24}-\frac{5\alpha}{4}-\frac{\beta}{8}\right)h^4y_i^{(4)} \\ +\left(\frac{-1}{16}+\frac{26\alpha}{48}+\frac{\beta}{48}\right)h^5y_i^{(5)}+\left(\frac{91}{5760}-\frac{82\alpha}{384}-\frac{\beta}{384}\right)h^6y_i^{(6)}+O(h^7);i=2,\dots\dots n-1\end{cases}$$

The scheme (2.10) – (2.12) gives rise to a family of methods of different orders as follows:

For $\alpha = \frac{1}{12}$ and $\beta = \frac{10}{12}$

2.1 Third order method

For $(w_0, w_1, w_2, w_3) = (24, -1, 1, 0)/d_1$ where $d_1 = 24$

Then the local truncation errors given by equation (2.16) are

$$t_i=\begin{cases}\frac{-247}{5760}h^5y_i^{(5)}+O\left(h^6\right),\ i=1,n \\ \frac{-1}{240}h^6y_i^{(6)}+O\left(h^7\right),\ i=2,3,\dots\dots,n-1\end{cases} \qquad (2.17)$$

2.2 Fourth order method

For $(w_0, w_1, w_2, w_3) = (6007, -981, 981, -247)/d_2$ *where* $d_2 = 5760$

Then the local truncation errors given by equation (2.16) are

$$t_i=\begin{cases}\frac{23}{576}h^6y_i^{(6)}+O\left(h^7\right),\ i=1,n \\ \frac{-1}{240}h^6y_i^{(6)}+O\left(h^7\right),\ i=2,3,\dots\dots,n-1\end{cases} \qquad (2.18)$$

Remark

(1) When $\alpha = \frac{1}{8}$ and $\beta = \frac{6}{8}$, then the scheme (2.10) is reduced to quadratic polynomial spline in [2, 3].

(2) When $\alpha = \frac{1}{24}$ and $\beta = \frac{22}{24}$, then the scheme (2.10) is reduced to cubic polynomial spline in [4]

(3) When $\alpha = \frac{1}{12}$ and $\beta = \frac{10}{12}$, we get new scheme that produces numerical results better than both quadratic and cubic polynomial splines in [2, 3, 4].

3. Spline solutions:

The spline solution of (1.1) with the boundary condition (1.2) is based on the linear equations given by (2.10) – (2.12), let

$Y=\left(y_{i+1/2}\right), S=\left(S_{i+1/2}\right), C=(C_i), T=(t_i)\,E=\left(e_{i+1/2}\right)=Y-S$

Be n-dimensional column vectors, then we can write the standard matrix equations for the nonpolynomial spline method in the form.

(i) $NY = C + T$

(ii) $NS = C$ (3.1)

(iii) $NE = T$

We also have $N = N_0 + h^2BF;\ F = \text{diag}\left(f_{i+1/2}\right)$ (3.2)

The three band symmetric matrix N_0 has the form:

$$N_0=\begin{pmatrix}-1 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -1\end{pmatrix} \qquad (3.3)$$

The matrix B has the form:

$$B=\begin{pmatrix}w_0 & w_1 & w_2 & w_3 & & & \\ \alpha & \beta & \alpha & & & & \\ & \alpha & \beta & \alpha & & & \\ & & & \ddots & & & \\ & & & \alpha & \beta & \alpha & \\ & & & & \alpha & \beta & \alpha \\ & & & w_3 & w_2 & w_1 & w_0\end{pmatrix} \qquad (3.4)$$

For the vector C, we have

$$C_i = \begin{cases} hA_1 + h^2\left[w_0 g_{1/2} + w_1 g_{3/2} + w_2 g_{5/2} + w_3 g_{7/2}\right], & i = 1 \\ h^2\left[\alpha g_{i-3/2} + \beta g_{i-1/2} + \alpha g_{i+1/2}\right], & i = 2,3,\ldots\ldots, n-1 \\ -hA_2 + h^2\left[w_0 g_{n-1/2} + w_1 g_{n-3/2} + w_2 g_{n-5/2} + w_3 g_{n-7/2}\right], & i = n \end{cases} \quad (3.5)$$

Set $N_0 = M_0 + J_0$ (3.6)

Where

$$M_0 = \begin{pmatrix} -3 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -3 \end{pmatrix} \quad (3.7)$$

And

$$J_0 = \begin{pmatrix} 2 & 0 & & 0 & 0 \\ 0 & 0 & & & 0 \\ \vdots & & & & \\ \vdots & & & & \\ 0 & & 0 & & \\ & & & 0 & 0 \\ 0 & 0 & & 0 & 2 \end{pmatrix} \quad (3.8)$$

4. Convergence analysis

Our main purpose now is to derive a bound on $\|E\|_\infty$. We now turn back to the error equation (*iii*) in (3.1) and rewrite it in the form

$$E = N^{-1}T = \left(M_0 + J_0 + h^2BF\right)^{-1}T = \left(I + M_0^{-1}\left(J_0 + h^2BF\right)\right)^{-1}M_0^{-1}T$$

This implies that

$$\|E\|_\infty = \left\|\left(I + M_0^{-1}\left(J_0 + h^2BF\right)\right)^{-1}\right\|_\infty \|M_0^{-1}\|_\infty \|T\|_\infty \quad (4.1)$$

In order to derive the bound on $\|E\|_\infty$, the following two lemmas are needed.

Lemma 4.1 ([10]). If G is a square matrix of order n and $\|G\| < 1$, then $(I+G)^{-1}$ exists and

$$\|(I+G)^{-1}\| < \frac{1}{1 - \|G\|}$$

Lemma 4.2; the matrix $\left(M_0 + J_0 + h^2BF\right)$ is nonsingular if $\|f\| < \dfrac{h^2 - 2w}{h^2 w(\beta + 2\alpha)}$ where

$$w = \frac{1}{8}\left((b-a)^2 + h^2\right)$$

Proof. Since,

$$N = \left(M_0 + J_0 + h^2BF\right) = \left(I + M_0^{-1}\left(J_0 + h^2BF\right)\right)M_0$$

and the matrix M_0 is nonsingular, so to prove N nonsingular it is sufficient to show $\left(I + M_0^{-1}\left(J_0 + h^2BF\right)\right)$ nonsingular.

Moreover,

$$\|F\|_\infty \le \|f\| = \max_{a \le x \le b} |f(x)| \quad (4.2)$$

$$\|M_0^{-1}\|_\infty \le \frac{h^{-2}}{8}\left[(b-a)^2 + h^2\right], \ see\ [6] \quad (4.3)$$

$$\|J_0\|_\infty = 2 \quad (4.4)$$

And $\|B\|_\infty = 2\alpha + \beta$ (4.5)

Also,

$$\left\|M_0^{-1}\left(J_0 + h^2BF\right)\right\|_\infty = \|M_0^{-1}\|_\infty \cdot \left(\|J_0\|_\infty + h^2\|B\|_\infty \|F\|_\infty\right) \quad (4.6)$$

Therefore, substituting $\|F\|, \|M_0^{-1}\|, \|J_0\|\ and\ \|B\|$ in (4.6) we get

$$\left\|M_0^{-1}\left(J_0 + h^2BF\right)\right\|_\infty \le \frac{h^{-2}}{8}\left((b-a)^2 + h^2\right)\left[2 + h^2(2\alpha+\beta)\|f\|\right] \quad (4.7)$$

Since, $\|f\| < \dfrac{h^2 - 2w}{h^2 w(\beta + 2\alpha)}$ (4.8)

Therefore, Eq. (4.8) leads to

$$\left\|M_0^{-1}\left(J_0 + h^2BF\right)\right\|_\infty \le 1 \quad (4.9)$$

From Lemma 4.1, it shows that the matrix N is nonsingular. Since , $\left\|M_0^{-1}\left(J_0 + h^2BF\right)\right\|_\infty < 1$ so using Lemma (4.1)and Eq. (4.1) follow that

$$\|E\|_\infty \le \frac{\|M_0^{-1}\|_\infty \|T\|_\infty}{1 - \|M_0^{-1}\|_\infty \|J_0 + h^2BF\|_\infty} \quad (4.10)$$

From Eq. (2.17) we have

$$\|T\|_\infty = \frac{247}{5760}h^5M_5; M_5 = \max_{a \le x \le b}\left|y^{(5)}(x)\right|$$

Then

$$\|E\|_\infty \le \frac{\|M_0^{-1}\|_\infty \|T\|_\infty}{1 - \|M_0^{-1}\|_\infty \|J_0 + h^2BF\|_\infty} \cong O\left(h^3\right) \quad (4.11)$$

Also, from Eq. (2.18) we have

$$\|T\|_\infty = \frac{23}{576}h^6M_6; M_6 = \max_{a \le x \le b}\left|y^{(6)}(x)\right|$$

Then

$$\|E\|_\infty \leq \frac{\|M_0^{-1}\|_\infty \|T\|_\infty}{1-\|M_0^{-1}\|_\infty \|J_0 + h^2 BF\|_\infty} \cong O(h^4) \quad (4.12)$$

We summarize the above results in the next theorem.

Theorem 4.1

Let *y(x)* is the exact solution of the continuous boundary value problem (1.1) with the boundary condition (1.2) and let $y_{i+1/2}, i = 0,1,\ldots.. n-1$, satisfies the discrete boundary value problem (*ii*) in (3.1). Further, if $e_{i+1/2} = y_{i+1/2} - S_{i+1/2}$ then

1- $\|E\|_\infty \cong O(h^3)$, for third order convergent method

2- $\|E\|_\infty \cong O(h^4)$, for fourth order convergent method

Which are given by (4.11) and (4.12), neglecting all errors due to round off.

5. Numerical examples and discussion:

We now consider two numerical examples illustrating the comparative performance of cubic nonpolynomial spline method (*ii*) in (3.1) over quadratic nonpolynomial spline method and the two polynomial spline methods (quadratic and cubic). All calculations are implemented by MATLAB 7

Example 1

Consider the boundary value problem, see [5]

$$y^{(2)} + y = -1 \quad (5.1)$$

$$y^{(1)}(0) = \frac{1-\cos(1)}{\sin(1)} = -y^{(1)}(1)$$

The analytical solution of (5.1) is

$$y(x) = \cos(x) + \frac{1-\cos(1)}{\sin(1)}\sin(x) - 1 \quad (5.2)$$

Example 2

Consider the boundary value problem, see [5]

$$y^{(2)} + xy = (3 - x - x^2 + x^3)\sin(x) + 4x\cos(x) \quad (5.3)$$

$$y^{(1)}(0) = -1,\ y^{(1)}(1) = 2\sin(1)$$

The analytical solution of (5.3) is

$$y(x) = (x^2 - 1)\sin(x) \quad (5.4)$$

The numerical results of examples 1 and 2 are presented in tables 1 and 2, respectively, for our fourth order method. A comparison between the method (2.10) and the existing methods in Ramadan et al. [5] are provided in tables 3 and 4.

Table 1: Approximate, Exact Solutions and Maximum errors (in absolute value) for Example 1 using our fourth order.

n	S_i **(approximated)**	y_i **(Exact)**	***E*** **(Error)**
4	0.13068504600377	0.13060321651340	8.18295–5[a]
8	0.13727099391989	0.13726907762415	1.91630–6
16	0.13893760135665	0.13893757908329	2.22734–8
32	0.00841355938534	0.00841356124929	1.86395–9
64	0.00423742716766	0.00423742736291	1.95255–10
128	0.00212635927486	0.00212635928910	1.42408–11

[a]8.18295–5 = $8.18295*10^{-5}$

Table 2: Approximate, Exact Solutions and Maximum errors (in absolute value) for Example 2 using our fourth order.

n	S_i **(approximated)**	y_i **(Exact)**	***E*** **(Error)**
4	- 0.35932989074946	- 0.35654365069809	2.78624–3
8	- 0.34264531263123	- 0.34258197359850	6.33390–5
16	- 0.29719707294621	- 0.29719640852255	6.64424–7
32	- 0.02582552960734	- 0.02582552960734	6.88566–8
64	- 0.01303052171412	- 0.01303052858255	6.86843–9
128	- 0.00654464520562	- 0.00654464571154	5.05929–10

Table 3: Maximum errors (in absolute value) for Example 1.

n	**Our fourth order method**	**Our third order method**	**Quadratic nonpoly. [5]**	**Cubic polyn. [5]**	**Quadratic polyn. [5]**
4	8.18295-5	8.18295-5	1.43181-3	2.85364-3	3.03488-3
8	1.91630-6	8.04854-6	1.75382-4	7.12633-4	7.69627-4
16	2.22734-8	5.91344-7	2.16003-5	1.78109-4	1.93094-4
32	1.86395-9	3.96416-8	2.67705-6	4.45241-5	4.83167-5
64	1.95255-10	2.56011-9	3.33110-7	1.11308-5	1.208186-5
128	1.42408-11	1.62945-10	4.15407-8	2.78270-6	3.02063-6

Table 4: Maximum errors (in absolute value) for Example 2.

n	Our fourth order method	Our third order method	Quadratic nonpoly. [5]	Cubic polyn. [5]	Quadratic polyn. [5]
4	2.78624-3	3.27323-3	2.2425-2	4.62182-2	4.94551-2
8	6.33390-5	3.03799-4	2.66946-3	1.15362-2	1.23088-2
16	6.64424-7	2.17464-5	3.24076-4	2.88302-3	3.08111-3
32	6.88566-8	1.43875-6	3.98761-5	7.20696-4	7.70391-4
64	6.86843-9	9.22972-8	4.94425-6	1.80171-4	1.92590-4
128	5.059294-10	5.84115-9	6.15517-7	4.50424-5	4.79946-5

6. Conclusion:

Two new methods are presented for solving second order two-point boundary value problem with Neumann conditions. These methods are shown to be optimal third and optimal fourth orders which are better than the two polynomial spline methods (quadratic and cubic splines) and quadratic nonpolynomial spline method. Moreover, nonpolynomial spline method has less computational cost over other polynomial spline methods. The obtained numerical results show that the proposed methods maintain a remarkable high accuracy which make them are very encouraging over other existing methods.

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7. References

1. A. Khan, Parametric cubic spline solution of two-point boundary value problems, Applied Mathematics and Computations 154 (2004) 175-182.
2. E.A.Al-Said, Spline solutions for system of second order boundary-value problems, int. J. Comput. Math 62 (1996) 143-154.
3. M.I. Hossam, S.Z. Sakr and W.K. Zahra, Quadratic spline solution to two point boundary value problems, Delta Journal of Science, 27 (2003) 4-14.
4. E.A. Al-Said, The use of cubic splines is the numerical solution of a system of second-order boundary value problems. Comput. Math. Applics 42 (2001) 861-869.
5. M.A. Ramadan, I. F. Lashien, W.K. Zahra, Polynomial and non-polynomial spline approaches to the numerical solution of second-order boundary-value problems, Applied Mathematics and Computations 184 (2007) 476-484.
6. D. Murray, Explicit inverses of Toeplitz and Associated Matrices, Anziam J. 44, 185-215, 2003.
7. W.K. Zahra, Numerical solutions for boundary value problems for ordinary differential equations by using spline functions, M.Sc. thesis, Faculty of Engineering, Tanta University, Egypt, 2004.
8. J.L. Blue, Spline function methods for nonlinear boundary value problems, communications of the ACM (1969).
9. C.V. Raghavarao, Y.V.S.S. Sanyasiraju, S. Suresh, A note on application of cubic splines to two-point boundary value problems, Comput. Math. Applics 27(11) (1994) 45-48.
10. R.A. Usmani, Discrete methods for a boundary value problem with engineering applications, Math. Comput. 32 (144) (1978) 1087-1096.

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