Disk-Rim flywheel of minimum weight

Bedier B. EL-Naggar and Ismail A. Kholeif
Department of Engineering Mathematics and Physics, Faculty of Engineering, Cairo University
Giza, Egypt
bbnaggar@hotmail.com

Abstract: In this article the disk-rim flywheel is suggested for light weight. The mass of the flywheel is minimized subject to constraints of required moment of inertia and admissible stresses. The theory of the rotating disks of uniform thickness and density is applied to each the disk and the rim independently with suitable matching condition at the junction. Suitable boundary conditions on the centrifugal stresses are applied and the dimensional ratios are obtained for minimum weight. It is proved that the required design is very close to the disk with uniform thickness.
[Bedier B. EL-Naggar and Ismail A. Kholeif. **Disk-Rim flywheel of minimum weight.** Journal of American Science 2011;7(6):146-149]. (ISSN: 1545-1003). http://www.americanscience.org.

Keywords: Disk-Rim; flywheel; minimum angular speeds; moment of inertia; radial and tangential stresses.

Introduction

Fluctuating power and/or load machines are equipped with a flywheel to store kinetic energy upon rotation. Strokes of large power increase the wheel rotation whereas this increased speed is reduced in strokes of low energy. If the energy fluctuation each cycle is ΔE , $\omega_{\max}$ and $\omega_{\min}$ are the maximum and minimum angular speeds; then $\Delta E = \frac{1}{2} I(\omega_{\max}^2 - \omega_{\min}^2)$. Here I is the mass moment of inertia of the flywheel about the axis of rotation. If the difference $\omega_{\max}^2 - \omega_{\min}^2$ is required not to exceed a given value in a certain application, the value of I is then fixed. Flywheels are then designed to ensure this value. Upon deciding the material of the flywheel the dimensions are determined accordingly. Usually the materials used are cast iron and steels.

The value of the maximum speed $\omega_{\max}$ is of primary importance in the design of the flywheels became higher speeds result in higher centrifugal stresses which should not exceed the admissible values of the flywheel material. This will be discussed later in detail. As modern designs require light weight, the design parameters are chosen according to ensure minimum weights with inertia and stresses are prescribed as constraints. As the moment of inertia of a mass element about a given axis is proportional to the square of the distance between the element and the axis, smaller mass at a large distance is more preferable than larger mass at small distances from the point of view of minimum weight. However in bodies of revolution larger distances imply larger circumference and areas. The two factors must be investigated properly the rim-disk wheel is suggested to ensure this. Figure (1) shows a schematic of disk-rim flywheel considered.

We should indicate that the flywheel is fitted with a hub around the axis of rotation for mounting the shaft. This hub serves as a reinforcement of the disc and dropping it in the calculation will be an approximation in the safe side.

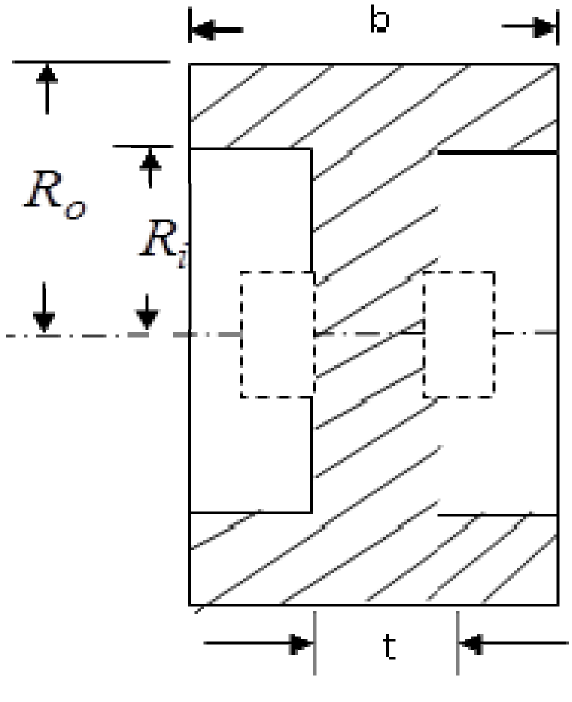

Figure 1

The mass M of the flywheel is given by

$$\frac{Mg}{\pi\gamma} = R_i^2\, t\left[1 + y\left(x^2 - 1\right)\right] \quad (1)$$

Here;

g acceleration of gravity

γ specific weight of flywheel material.

R_i disk radius

t disk thickness

$x = \frac{R_0}{R_i}$; R_0 outer radius of rim

$y = \frac{b}{t}$, b width of rim

Also, the moment of inertia I is given by

$$\frac{2Ig}{\pi\gamma} = R_i^4\, t\left[1 + y\left(x^4 - 1\right)\right] \quad (2)$$

Different aspects of flywheel design are investigated by several authors along with other rotating disk machine elements. You et. al.[1] made numerical analysis of elastic plastic rotating disks with arbitrary variable thickness and density; the governing equation is derived from the basic equations of rotating disks and solved using the Runge-Kutta Algorithm. Also, Sterner et. al.[2] developed a unified numerical approach for the analysis of rotating disks including turbine rotors. The problem of stresses in linearly hardening rotating solid disks of variable thickness is discussed by Orcan and Eraslan[3], analytically flywheels with friction used for optimal control of damping are studied by EL-Gohary [4]. The problem of robust stabilization and robust output feedback stabilization of the angular velocity of a rigid body are tackled by Astofi A. [5,6], the problem of rotating anisotropic disk of uniform strength is investigated by Jain et al.[7]. Callioglu [8] made an analysis of the stresses in an orthotropic rotating disk under thermal loading. The problem of limit angular velocities of variable thickness rotating disks is solved by Eraslan et al. [9].

Formulation and Solution:

Consider the disk-rim flywheel whose cross-section is shown in figure (1). The flywheel is rotating around its axis of symmetry at an angular speed ω, the radial and tangential stresses σ_r and σ_θ respectively are given by [10]:

$$\sigma_r = \frac{A}{2} - \frac{3+\upsilon}{8}\frac{\gamma\omega^2}{g} r^2 + \frac{B}{r^2} \quad (3)$$

And $$\sigma_\theta = \frac{A}{2} - \frac{1+3\upsilon}{8}\frac{\gamma\omega^2}{g} r^2 + \frac{B}{r^2} \quad (4)$$

here υ is the Poisson's ratio of the material A and B are arbitrary constants to be determined from the boundary conditions imposed on both the disk and the rim.

Stresses in the disk:

The disk of Radius R_i and thickness t includes the centre line $r = 0$. For bounded values of stresses the constant B must be chosen zero and the resulting stresses in the disk are reduced to:

$$\sigma_r\big|_{disk} = \frac{A}{2} - \frac{3+\upsilon}{8}\frac{\gamma\omega^2}{g} r^2 \quad (5)$$

And $$\sigma_\theta\big|_{disk} = \frac{A}{2} - \frac{1+3\upsilon}{8}\frac{\gamma\omega^2}{g} r^2 \quad (6)$$

the maximum value of this stresses occur at the centre line and are equal to the yet undetermined arbitrary constants $\frac{A}{2}$ to be determined from the boundary condition at $r = R_i$ where the disk and the rim join together. This condition is imposed by the centrifugal force on the rim transmitted to the edge of the disk as radial stresses $\sigma_r\big|_{R_i}$. The total centrifugal force on the rim is given by $M_{rim}\,\omega^2\,\overline{R}_{rim}$ where M_{rim} is the mass of the rim and $\overline{R}_{rim}$ is the mean radius of the rim. We have $M_{rim} = \frac{\gamma}{g}\pi b\left(R_0^2 - R_i^2\right)$ and $\overline{R}_{rim} = \frac{1}{2}\left(R_i + R_0\right)$.

The area over which this force is uniformly distributed is the contact area between the disk and the rim $2\pi R_i t$. The resulting radial stress on the disk edge σ_r radial $\big|_{R_i}$ is given by:

$$\frac{1}{4}\frac{\gamma\omega^2}{g} R_i^2\, y\left(x^2 - 1\right)\left(x + 1\right) ,$$ x and y are defined before, substituting this value in equation (5) for $r = R_i$ gives

$$\frac{A}{2}=\sigma_{r\ \text{diskmax}}=\sigma_{\theta\ \text{diskmax}}=\frac{\gamma\omega^2}{g}R_i^2\left[\begin{array}{l}\frac{3+\upsilon}{8}\\ +\frac{1}{4}y\left(x^2-1\right)\left(x+1\right)\end{array}\right] \quad (7)$$

this value should be equal to the admissible stress of the material σ for safe design. Accordingly, the second constraint on design is given by:

$$\frac{\sigma g}{\gamma\omega^2}=R_i^2\left[\frac{3+\upsilon}{8}+\frac{1}{4}y\left(x^2-1\right)\left(x+1\right)\right] \quad (8)$$

the first constraint is that the moment of inertia is given by equation (2).

Stresses in the rim

In the rim, the centre line $r=0$ is not included and the arbitrary constant B can not be dropped. Accordingly, the stress distribution on the rim is given by equations (3) and (4). A relation between the values of the constants A and B in the rim can be obtained by applying the condition $\sigma_{r\ \text{rim.}}\big|_{R_0}=0$ since the edge R_0 of the rim is free from external radial stresses. In terms of the constant A - not yet determined the expressions of the stresses in the rim are:

$$\sigma_r=\frac{A}{2}\left(1-\frac{R_0^2}{r^2}\right)+\frac{3+\upsilon}{8}\frac{\gamma\omega^2}{g}\left(\frac{R_0^4}{r^2}-r^2\right) \quad (9)$$

$$\sigma_\theta=\frac{A}{2}\left(1-\frac{R_0^2}{r^2}\right)+\frac{\gamma\omega^2}{g}\left(\frac{3+\upsilon}{8}\frac{R_0^4}{r^2}-\frac{1+3\upsilon}{8}r^2\right) \quad (10)$$

The value of the constant $\frac{A}{2}$ is determined from the value of centrifugal force on the rim divided by the area of the rim at the edge giving

$$\sigma_{r\ \text{rim}}\big|_{R_i}=\frac{\gamma\omega^2}{4g}R_i^2\left(x^2-1\right)\left(x+1\right) \quad (11)$$

Equations (9) and (11) give:

$$\frac{A}{2}=\frac{\gamma\omega^2}{g}R_i^2\left[\frac{3+\upsilon}{8}\left(x^2+1\right)-\frac{1}{4}\left(x+1\right)\right] \quad (12)$$

Using equation (12) in equations (9) & (10) give:

$$\sigma_{r\ \text{rim}}=\frac{\gamma\omega^2}{g}R_i^2\left[\frac{3+\upsilon}{8}\left(x^2+1\right)-\frac{1}{4}\left(x+1\right)\right]\left(1-\frac{R_0^2}{r^2}\right)+\frac{\gamma\omega^2}{g}\frac{3+\upsilon}{8}\left(\frac{R_0^4}{r^2}-r^2\right) \quad (13)$$

$$\sigma_{\theta\ \text{rim}}=\frac{\gamma\omega^2}{g}R_i^2\left[\begin{array}{l}\frac{1}{4}\left(x+1\right)\left(1-x^2\frac{R_i^2}{r^2}\right)+\\ \frac{3+\upsilon}{8}\left(x^2+1\right)\left(x^2\frac{R_i^2}{r^2}-1\right)-\frac{1+3\upsilon}{8}\frac{r^2}{R_i^2}\end{array}\right] \quad \textbf{(14)}$$

Investigation of equation (14) for maximum shows that the maximum occurs for

$$\left(\frac{r}{R_i}\right)^4=\frac{2\left(x+1\right)-\left(3+\upsilon\right)\left(x^2-1\right)}{1+3\upsilon}x^2$$

for $x\geq 1$ this gives negative r and therefore rejected. The accepted value for maximum $\sigma_{\theta\ \text{rim}}$ occurs at $r=R_i$ and this gives the third constraint on the minimization of the flywheel mass as:

$$\frac{\sigma g}{\gamma\omega^2}=R_i^2\left[\frac{1}{4}\left(1-x^2\right)\left(x+1\right)+\frac{3+\upsilon}{8}\left(x^4-1\right)-\frac{1+3\upsilon}{8}\right] \quad (15)$$

Equations (8) and (15) give

$$y=\frac{\frac{3+\upsilon}{8}\left(x^4-2\right)-\frac{1}{4}\left(x^2-1\right)\left(x+1\right)-\frac{1+3\upsilon}{8}}{\left(x^2-1\right)\left(x+1\right)} \quad (16)$$

and $R_i^2=\dfrac{\sigma g}{\gamma\omega^2}\dfrac{1}{\frac{3+\upsilon}{8}+\frac{1}{4}\left(x^2-1\right)\left(x+1\right)y}$ (17)

where y is given in terms of x in equation (16). Having determined R_i^4 and y as functions of x, the x dependence of t is given by :

$$t=\frac{2}{\pi}\frac{I\gamma g\omega^4}{\sigma^2}\frac{\left[\frac{3+\upsilon}{8}\left(x^4-1\right)-\frac{1}{4}\left(x^2-1\right)\left(x+1\right)-\frac{1+3\upsilon}{8}\right]^2}{1+\frac{x^2+1}{x+1}\left[\frac{3+\upsilon}{8}\left(x^4-2\right)-\frac{1}{4}\left(x^2-1\right)\left(x+1\right)-\frac{1+3\upsilon}{8}\right]} \quad (18)$$

Inserting values of R_i^2, t and y as a function of x in equation (1), we require to minimize the function:

$$\frac{\left[\frac{3+\upsilon}{8}\left(x^4-1\right)-\frac{1}{4}\left(x^2-1\right)\left(x+1\right)-\frac{1+3\upsilon}{8}\right]\times 1+\frac{1}{x+1}\left[\frac{3+\upsilon}{8}\left(x^4-2\right)-\frac{1}{4}\left(x^2-1\right)\left(x+1\right)-\frac{1+3\upsilon}{8}\right]}{1+\frac{x^2+1}{x+1}\left[\frac{3+\upsilon}{8}\left(x^4-2\right)-\frac{1}{4}\left(x^2-1\right)\left(x+1\right)-\frac{1+3\upsilon}{8}\right]} \quad (19)$$

The minimum for $\upsilon=0.3$ is found to be very close to $x=4$ where

$$y=1.144, \qquad R_i=0.11\sqrt{\frac{\sigma g}{\gamma\omega^2}},$$

$$t=(24.3)\left(\frac{2}{\pi}\right)\frac{I\gamma\omega^4}{g\sigma^2},\ M=5.16\left(\frac{2}{\pi}\right)\frac{I\gamma\omega^2}{g\sigma^2},$$

$$R_0=0.44\sqrt{\frac{\sigma g}{\gamma\omega^2}} \quad \text{and} \quad b=(27.8)\left(\frac{2}{\pi}\right)\frac{I\gamma\omega^4}{g\sigma^2}$$

Conclusion:

The result of this analysis is remarkable; the width of the rim is approximately equal to the thickness of the disk. Besides most of the side area of the flywheel is spanned by the width of the rim the flywheel tends to be close to the uniform disk of constant thickness. This is for minimum weight. It became obvious that wide rims are used in rim flywheels when the disk is replaced by a set of long arms joining it to the hub.

.

References

1. You, L. H., Tang Y. Y. , Zhang C. Y. " Numerical analysis of elastic-plastic rotating disks with arbitrary variable thickness and density"; Int. J. of solids and structures,(2000)vol. 37 pp7809-7820
2. Sterner S. C " A unified numerical approach for the analysis of rotating disks including turbine rotors", Int. J. of solids and structures,(1993)vol. 31 pp269-277
3. Orcan Y. & Eraslan A., "Elastic-plastic stresses in linearly hardening rotating solid disks of variable thickness", Mechanics research commun.,(2002), vol.29 pp.269-281.
4. EL-Gohary, A." Optimal control of damping rotation of a rigid body using flywheel with friction", Chaos, solitons and fracts, (2005), vol. 24, pp. 207-221
5. Astofi A. and Rapaport A. " Robust stabilization of the angular velocity of a rigid body", syst. control lett. (1998); vol. 34, pp. 257-264
6. Mazenc F., Astofi A. " Robust output feedback stabilization of the angular velocity of a rigid body" syst. control lett. (2000); vol. 39, pp. 203-210
7. Jain R, Ramachandra K., Simha KRY. " Rotating anisotropic disk of uniform strength" Int. J. of Mech. Science(1999), vol. 41, pp 639-48.
8. Cullioglue H. "Stress analysis of an orthotropic rotating disk under thermal loading" J. of Reinforced plastics and composites(2004), vol. 23(17) pp. 1859-1867.
9. Eraslan AN, Argeso H. " Limit angular velocities of variable thickness rotating disks", Int. J. of Solids and structures (2002), vol. 39, pp. 3109-3130
10. Erich Hau and Horst Von Renouard, Wind Turbines(chapter 6), 2006, Springer Verlag, Berlin.

5/22/2011