Theoretical Analysis in Particle Swarm Optimisation

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Abstract: Nowadays, particle swarm optimisation (PSO) is considered as one of the most efficient techniques in solving optimisation problems. Implementing theoretical analyses in PSO can lead to a more deep understanding of its behavior and characteristics and may quicken the improvement in its computational behavior. However, almost all the existing findings on PSO are based on experimental observations and just in a few cases, theoretical analysis on PSO has been implemented. In this paper, theoretical analyses implemented on PSO are reviewed and some directions for future research are proposed.

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1. Introduction

There are so many optimisation problems in various areas of science and engineering. For solving them, there exist twofold approaches; classical approaches and heuristic approaches. Classical approaches such as linear programming and non-linear programming are not efficient enough in solving optimisation problems. Since they suffer from curse of dimensionality and also require preconditions such as continuity and differentiability of objective function that usually are not met.

Heuristic approaches which are usually bio-inspired include a lot of approaches such as genetic algorithms, evolution strategies, differential evolution and so on. Heuristics do not expose most of the drawbacks of classical and technical approaches. Among heuristics, particle swarm optimisation (PSO) has shown more promising behavior.

PSO is a stochastic, population-based optimisation technique introduced by Kennedy and Eberhart (Kennedy and Eberhart, 1995). It belongs to the family of swarm intelligence computational techniques and is inspired of social interaction in human beings and animals (especially bird flocking and fish schooling).

Some PSO features that make it so efficient in solving optimisation problems are the followings:

- In comparison with other heuristics, it has less parameters to be tuned by user.
- Its underlying concepts are so simple. Also its coding is so easy.
- It provides fast convergence.
- It requires less computational burden in comparison with most other heuristics.
- It provides high accuracy.
- Roughly, initial solutions do not affect its computational behavior.
- Its behavior is not highly affected by increase in dimensionality.
- It is efficient in tackling multi-objectives, multi-modalities, constraints, discrete/integer variables.
- There exist many efficient strategies in PSO for mitigating "premature convergence." Thus, its success rate is so high.

Despite the fact that implementing theoretical analyses in PSO can lead to a more deep understanding of its behavior and characteristics and may quicken the improvement in its computational behavior, just in a few cases, theoretical analyses on PSO have been conducted. It is undeniable that this aspect should be paid more attention by PSO research community. In this paper, the aim is to review all the implemented theoretical analyses on PSO and propose some directions for future research.

The paper is organised as follows; in section 2, an overview of PSO is provided. In section 3, all the implemented theoretical analyses on PSO are reviewed. Finally, drawing conclusions and proposing some directions for future research is implemented in section 4.

2. PSO Overview

PSO starts with the random initialisation of a population (swarm) of individuals (particles) in the n-dimensional search space (n is the dimension of problem in hand). The particles fly over search space with adjusted velocities. In PSO, each particle keeps two values in its memory; its own best experience, that is, the one with the best fitness value (best fitness value corresponds to least objective value since fitness function is conversely proportional to objective function) whose position and objective value are called P_i and P_{best} respectively and the best experience of the whole swarm, whose position and objective value are called P_g and g_{best} respectively. Let denote the position and velocity of particle i with the following vectors:

$$X_i = (X_{i1}, X_{i2}, \dots, X_{id}, \dots, X_{in})$$

$$V_i = (V_{i1}, V_{i2}, \dots, V_{id}, \dots, V_{in})$$

The velocities and positions of particles are updated in each time step according to the following equations:

$$V_{id}(t+1) = V_{id}(t) + C_1 r_{1d}(P_{id} - X_{id}) + C_2 r_{2d}(P_{gd} - X_{id}) \quad (1)$$

$$X_{id}(t+1) = X_{id}(t) + V_{id}(t+1) \quad (2)$$

Where C_1 and C_2 are two positive numbers and r_{1d} and r_{2d} are two random numbers with uniform distribution in the interval [0,1]. Here, according to (1), there are three following terms in velocity update equation:

1) The first term this models the tendency of a particle to remain in the same direction it has traversing and is called "inertia," "habit," or "momentum."
2) The second term is a linear attraction toward the particle's own best experience scaled by a random weight $C_1 r_{1d}$. This term is called "memory," "nostalgia," or "self-knowledge."
3) The third term is a linear attraction toward the best experience of the all particles in the swarm, scaled by a random weight $C_2 r_{2d}$. This term is called "cooperation," "shared information," or "social knowledge."

The procedure for implementation of PSO is as follows:

1) Particles' velocities and positions are Initialised randomly, the objective value of all particles are calculated, the position and objective of each particle are set as its P_i and P_{best} respectively and also the position and objective of the particle with the best fitness (least objective) is set as P_g and g_{best} respectively.
2) Particles' velocities and positions are updated according to equations (1) and (2).
3) Each particle's P_{best} and P_i are updated, that is, if the current fitness of the particle is better than its P_{best}, P_{best} and P_i are replaced with current objective value and position vector respectively.
4) P_g and g_{best} are updated, that is, if the current best fitness of the whole swarm is fitter than g_{best}, g_{best} and P_g are replaced with current best objective and its corresponding position vector respectively.
5) Steps 2-4 are repeated until stopping criterion (usually a prespecified number of iterations or a quality threshold for objective value) is reached.

It should be mentioned that since the velocity update equations are stochastic, the velocities may become too high, so that the particles become uncontrolled and exceed search space. Therefore, velocities are bounded to a maximum value V_{max}, that is (Eberhart, Shi and Kennedy, 2001):

$$\text{If } |V_{id}| > V_{max} \; then \; V_{id} = \text{sign}(V_{id}) V_{max} \quad (3)$$

Where sign represents sign function.

However, primary PSO characterised by (1) and (2) does not work desirably; especially since it possess no strategy for adjusting the trade-off between explorative and exploitative capabilities of PSO. Therefore, the inertia weight PSO is introduced to remove this drawback. In inertia-weight PSO, which is the most commonly-used PSO variant, the velocities of particles in previous time step is multiplied by a parameter called inertia weight. The corresponding velocity update equations are as follows (Shi and Eberhart, 1998), (Shi and Eberhart, 1999):

$$V_{id}(t+1) = \omega V_{id}(t) + C_1 r_{1d}(P_i - X_{id}) + C_2 r_{2d}(P_{gd} - X_{id})$$

$$X_{id}(t+1) = X_{id}(t) + V_{id}(t+1) \quad (4)$$

Inertia weight adjusts the trade-off between exploration and exploitation capabilities of PSO. The less the inertia weight is, the more the exploration capability of PSO will be and vice versa. Commonly, it is decreased linearly during the course of the run, so that the search effort is mainly focused on exploration at initial stages and is focused more on exploitation at latter stages of the run.

3. Theoretical Analyses in PSO

Almost all the existing findings and conclusions on PSO are based on experimental observations. However, implementing theoretical analyses in PSO can lead to a more deep understanding of its behavior and characteristics and may quicken the improvement in its computational behavior. But due to some reasons, the implementation of theoretical analysis is so difficult and has been rarely undertaken in PSO literature. Some of the reasons are as proceed.

- The forces among particles are stochastic that hinders utilising standard mathematical tools used in the analysis of dynamic systems.
- The PSO's performance is strongly dependent on the used fitness function, whereas there exist so

many different types of fitness functions. Consequently, drawing conclusions applicable to all type of fitness functions is so challenging.

- In PSO, the swarm consists of a large number of particles which understanding the dynamic of the whole is difficult.
- The inherent stochasticity existent is a crucial factor in PSO that makes its analysis so difficult.
- The memory that particles and swarm possess, makes the analysis more challenging.

Due to the above-mentioned reasons, some simplifying assumptions are used while conducting theoretical analyses in PSO. Some commonly used simplifying assumptions are ignorance of stochasticity, assuming swarm as a single isolated particle in one-dimension, assuming the coincidence of personal and neighborhood best, assuming search process in stagnation state, and also ignoring velocity clamping and inertia weight. Needless to say, the less the simplifying assumptions are, the more is the validity and accuracy of drawn conclusions. So, researchers are attempting to model PSO's characteristics as real as possible. In this paper, theoretical analyses undertaken in specialised literature are classified into two main groups; those which do not consider the stochasticity of PSO and those who consider it, which are called deterministic-based (Ozcan and Mohan, 1998), (Shi and Eberhart, 2001), (Clerc and Kennedy, 2002), (Van den Bergh, 2002), (Brandstatter and Baumgartner, 2002), (Trelea, 2003), (Liang and Suganthan, 2005), (Yasuda and Iwasaki, 2003), (Blackwell, 2005), (Blackwell, 2003), (Campana, Fasano and Pinto, 2006), (Campana, Peri and Pinto, 2006), (Zhao, and Mao, 2009), (Samal, Konar, 2007), (Bratton, and Blackwell, 2007) and stochastic-based analyses (Clerc, 2006), (Kadirkamanathan, Selvarajah, and Fleming, 2006), (Poli, et. , al, 2007), (Poli, and Broomhead, 2007), (Poli, 2007), (Zhang, Li, Zhao, and Wang, 2009), (Helwig, and Wanka, 2007), (Chen, and Jiang, 2010), (Ghosh, et. , al, 2012), (Pena, 2008), (Helwig and Wanka, 2008), (Veeramachaneni, Osadciw and Kamath, 2007). They will be explained bellow.

3.1 Deterministic-based Theoretical Analyses in PSO

In the first attempt for theoretical analysis in PSO, in a highly approximate paradigm, PSO has been modeled as a single, isolated, one-dimension particle in stagnation state, while inertia weight, velocity clamping are ignored. According to this modeling, the particle's trajectories were determined (Ozcan and Mohan, 1998). This work was extended by modeling multiple particles in multi-dimension while P_i and P_g are not assumed to coincide (Shi and Eberhart, 2001). According to this modeling, it is concluded that the particles' trajectories change with C_1 and C_2 and the trajectories rely upon $C = C_1 + C_2$. Under the same simplifying assumptions with (Shi and Eberhart, 2001), later on, the swarm was considered as a discrete-time linear dynamic system wherein the dynamics of the state of the particles are determined by finding eigenvalues and eigenvectors of the state transition matrix and it is concluded that the particles converge to equilibrium if and only if the eigenvalues are less than unity. Consequently, since eigenvalues are functions of PSO parameters, the parameters guaranteeing PSO's convergence are determined (Clerc and Kennedy, 2002).

In a later work, under the same assumptions with (Shi and Eberhart, 2001), it is concluded that the particles are pulled toward the weighted some of personal and neighborhood best (Van den Bergh, 2002).

In (Brandstatter and Baumgartner, 2002), an analogy between PSO considered as a one-dimension single particle in stagnation state and a damped mass-spring oscillator is drawn, and notions of "damping factor" and vibrational frequency" were invoked to draw some guidelines for setting PSO's parameters.

In (Trelea, 2003), in a work similar to (Clerc and Kennedy, 2002) and under the same assumptions with (Ozcan and Mohan, 1998), the dynamic behavior and the convergence of PSO are analysed with discrete time dynamic system theory. Consequently, some guidelines for setting PSO's parameters are extracted.

In (Liang and Suganthan, 2005) and (Yasuda and Iwasaki, 2003), under the same assumptions with (Ozcan and Mohan, 1998), but considering inertia weight, eigenvalue analysis is invoked for determining PSO parameters which result in a stable dynamic system.

In (Blackwell, 2005), (Blackwell, 2003) the constricted PSO is modelled by a single multi-dimensional particle and changes in spatial extent of a particle is explored over time and it is concluded that spatial extent is diminished exponentially over time.

In (Campana, Fasano and Pinto, 2006) and (Campana, Peri and Pinto, 2006) fully informed particle swarm (FIPS) under the same assumptions with (Ozcan and Mohan, 1998) is modeled as a dynamic system and eigenvalue analysis is invoked to determine that each setting for PSO parameters leads to which type of computational behavior. Moreover, some guidelines for initialising positions and velocities are provided so that the most orthogonality in particles' trajectories is guaranteed.

In (Zhao, and Mao, 2009), PSO flight equations are transformed into a linear difference equation and the particle's stability criteria as a function of PSO parameters are derived.

In (Samal and Konar, 2007) a closed-loop stability analysis of PSO dynamics is implemented by Jury's test and root-locus technique. By Jury's stability test proposes some settings for PSO parameters. An explicit modelling of nonlinearity in feedback path is presented. In this analysis, unlike previous analyses that combined acceleration coefficients in one term, their separate existence is considered and their suitable range for achieving stability is determined. In table 1.1 all different deterministic-based theoretical analysis in PSO, their pros, cons and their main findings are tabulated.

3.2 Stochastic-based Theoretical Analyses in PSO

This group of analyses model the inherent stochasticity of PSO. However, there exists so limited number of these analyses in the literature.

In (Clerc, 2006), assuming a single particle in one dimension, during stagnation with considering the stochasticity of PSO, the distribution of particle velocity is analysed. It is showed that particle's new velocity is the combination of three following terms.

$$V(t+1) = K_1V(t) - K_2V(t-1) + K_3(P_i - P_g) \quad (5)$$

Where $K_1V(t)$ represents a forward force, $-K_2V(t-1)$ represents a backward force and also $K_3(P_i - P_g)$ indicates the noise. K_1, K_2 and K_3 are stochastic variables whose distribution models are determined. These distribution models depend on PSO parameters. It is proved that $E(K_1) = \omega - C^2.\mathrm{Ln}(2)$ and $E(K_2) = 2\omega.\mathrm{Ln}(2)$, where ω and C represent inertia weight and sum of acceleration coefficients respectively. By manipulating the distributions of K_1, K_2 and K_3 some conclusions and guidelines has been drawn.

In (Kadirkamanathan, Selvarajah, and Fleming, 2006), PSO is modeled as a single, one-dimension particle in stagnation state in g_{best} PSO with inertia weight considering the stochasticity and lyupanov stability analysis is invoked to investigate particle's stability. The particle is represented as a nonlinear feedback system, its transfer function is determined and its observability and controllability are proved. A lyupanov function for the system is determined and sufficient conditions on PSO parameters to guarantee convergence are derived. However, the derived conditions are so restrictive due to the conservative characteristic of lyupanov function.

In (Poli, et. , al, 2007), a discrete markov chain model of bare-bones PSO assuming a single one-dimensional particle is devised that can approximate it on arbitrary continuous problems to any precision. The objective function is discretised using finite element grid which produces corresponding distinct states in algorithm. Iterating the transfer matrix gives precise information about the behavior of optimizer at each iteration. The experiments strongly support the findings of this theoretical analysis.

In (Poli, and Broomhead, 2007) and (Poli, 2007), considering all characteristics of real PSO but assuming it in a stagnation state, firstly, the exact dynamic equations for the moments of sampling distribution are devised, then according to the conducted statistical analysis, areas in parameter space leading to stability are identified.

In (Zhang, Li, Zhao, and Wang, 2009), based on dynamic characteristic analysis of eigenvalues in Z-plane and control theory rules, new guidelines for setting PSO parameters are provided. The assumptions are a single one-dimensional particle in stagnation state.

In (Helwig, and Wanka, 2007) g_{best} PSO is modelled in high-dimensional constrained search spaces considering the stochasticity. Consequently, according the theoretical analysis, best fly back strategy for handling constraints in different type of problems are put forward. For modelling, PSO is modelled as multiple multi-dimensional particles.

In (Chen, and Jiang, 2010), the PSO's particle interaction behavior is analysed. Firstly, a statistical interpretation of PSO is provided in order to capture the stochastic behavior of the whole swarm. Based on statistical interpretation, the effect of particle interaction is investigated by focusing on social-only PSO model and the lower bounds of the expected particle norm are derived. The assumptions of this analysis are multiple particles, one-dimension, stochastic acceleration coefficients.

In (Ghosh, et. , al, 2012) PSO has been modeled with multiple one-dimensional particles and $l$best neighborhood. A state-space model of the swarm dynamics is presented and the necessary conditions that assure stability and asymptotic convergence of the particles' dynamics are determined by Jury and Blanchard's stability tests. Its salient aspect is modelling $l$best neighborhood topology which had not conducted before. In table 1.2 all different stochastic-based theoretical analysis in PSO, their pros, cons and their main findings are tabulated.

Table 1.1: Different deterministic-based theoretical analysis in PSO, their pros, cons and main findings

Reference Number	Modeling Assumptions	Main findings
[5]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are assumed to coincide.	The particle's trajectories were determined
[6]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are not assumed to coincide.	The particles' trajectories change with C_1 and C_2 and the trajectories rely upon $C = C_1 + C_2$.
[7]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are not assumed to coincide.	Using eigenvalue analysis, the PSO parameters guaranteeing its s convergence are determined.
[8]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are not assumed to coincide.	It is concluded that the particles are pulled toward the weighted some of personal and neighborhood best.
[9]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are not assumed to coincide.	Using the analogy between PSO and a damped mass-spring oscillator, some guidelines for setting PSO's parameters are derived.
[10]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are assumed to coincide.	Using discrete time dynamic system theory, some guidelines for setting PSO's parameters are extracted.
[11]- [12]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are assumed to coincide, but considering inertia weight.	Eigenvalue analysis is invoked for determining PSO parameters which result in a stable dynamic system.
[13]-[14]	Constricted PSO by a single multi-dimensional particle.	The changes in spatial extent of a particle is explored over time and it is concluded that spatial extent is diminished exponentially over time.
[15]-[16]	Fully informed particle swarm (FIPS), single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are assumed to coincide.	Eigenvalue analysis is invoked to determine that each setting for PSO parameters leads to which type of computational behavior. Moreover, some guidelines for initialising positions and velocities are provided so that the most orthogonality in particles' trajectories is guaranteed.
[17]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are assumed to coincide.	By solving linear difference equation, particle's stability criteria are derived as a function of PSO parameters.
[18]	Single, isolated, one-dimension particle in stagnation state, without stochasticity. P_i and P_g are assumed to coincide. But unlike previous analyses that combined acceleration coefficients in one term, their separate existence is considered.	A closed-loop stability analysis of PSO dynamics is implemented by Jury's test and root-locus technique. Jury's stability test proposes some settings for PSO parameters. An explicit modelling of nonlinearity in feedback path is presented. Also, suitable range of C_1 and C_2 for achieving stability is determined.

Table 1.2: Different stochastic-based theoretical analysis in PSO, their pros, cons and main findings

Reference Number	Modeling Assumptions	Main findings
[20]	A single particle in one dimension, during stagnation with considering the stochasticity.	The distribution of particle velocity is analysed. It is showed that particle's new velocity is the combination of three stochastic terms; a forward force, a backward force and a noise. The distribution models of static variables are determined. Consequently, some guidelines for tuning PSO parameters are provided.
[21]	A single, one-dimension particle in stagnation state in g_{best} PSO with inertia weight considering the stochasticity.	Lyupanov stability analysis is invoked to investigate particle's stability. The particle is represented as a nonlinear feedback system, its transfer function is determined and its observability and controllability are proved. Sufficient conditions on PSO parameters to guarantee convergence are derived.
[22]	Bare-bones PSO assuming a single one-dimensional particle.	A discrete markov chain model of bare-bones PSO is devised that can approximate it on arbitrary continuous problems to any precision. The objective function is discretised using finite element grid which produces corresponding distinct states in algorithm. Iterating the transfer matrix gives precise information about the behavior of optimizer at each iteration.
[23]-[24]	All characteristics of real PSO but assuming it in a stagnation state.	Firstly, the exact dynamic equations for the moments of sampling distribution are devised, then according to the conducted statistical analysis, areas in parameter space leading to stability are identified.
[25]	Single one-dimensional particle in stagnation state.	Based on dynamic characteristic analysis of eigenvalues in Z-plane and control theory rules, new guidelines for setting PSO parameters are provided.
[26]	g_{best} neighborhood, multiple high-dimensional particles in constrained search space, considering the stochasticity.	According the theoretical analysis, best fly back strategy for handling constraints in different type of problems are put forward.
[27]	Multiple particles, one-dimension, stochastic acceleration coefficients, considering interaction among particles, ignoring cognitive part of flight equation.	PSO's particle interaction behavior is analysed. Firstly, a statistical interpretation of PSO is provided. Based on it, the effect of particle interaction is investigated by focusing on social-only PSO model and the lower bounds of the expected particle norm are derived.
[28]	Multiple one-dimensional particles and l best neighborhood. The salient aspect is modelling l best neighborhood topology which had not conducted before.	A state-space model of the swarm dynamics is presented and the necessary conditions that assure stability and asymptotic convergence of the particles' dynamics are determined.

4. Conclusions and Future Research Directions

In this paper, theoretical analyses implemented on PSO are reviewed. The aim of theoretical analysis is to model PSO as real as possible. But, to date, this aim has not been approached as desired. A few researches have been done on theoretical analysis in PSO and it should be paid more attention. In particular, the following aspects are highlighted as promising directions for future research on this area.

- Devising more efficient approaches for modelling stochasticity of PSO.
- Implementing theoretical analysis with assuming swarm in stagnation state.
- Implementing theoretical analysis on PSO with different neighborhood topologies like wheel, ring,…, etc.
- Implementing theoretical analysis on PSO with multi-objective objective function.
- Implementing theoretical analysis on PSO with multi-modal objective function.
- Implementing theoretical analysis on PSO with dynamic objective function.
- Implementing theoretical analysis on PSO in constrained environments.
- Implementing theoretical analysis on PSO with discrete/binary/integer variables.
- Implementing theoretical analysis on computational time of PSO.

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References

1. Kennedy J, Eberhart R. Particle swarm optimisation. Proc. IEEE Int. Conf. on Neural Networks, 1995: 1942-8.
2. Eberhart R, Shi Y, Kennedy J. Swarm Intelligence, Morgan Kaufmann, San Mateo, 2001.
3. Shi Y, Eberhart R. A modified Particle swarm optimizer. Proc. IEEE Int. Conf. on Computational Intelligence, 1998: 69-73.
4. Shi Y, Eberhart R. Empirical study of particle swarm optimisation. Proc. IEEE Int. Conf. on Computational Intelligence, 1999: 1945-50.
5. Ozcan E. Mohan CK, Analysis of a simple particle swarm optimization algorithm. Intelligent Engineering Systems Through Artificial Neural Networks 1998;8:253-8.
6. Shi Y, Eberhart R. Fuzzy adaptive particle swarm optimisation. Proc. IEEE Int. Congress on Evolutionary Computation, 2001: 101-6.
7. Clerc M, Kennedy J, The particle swarm explosion, stability and convergence in a multi-dimensional complex space. IEEE Trans. Evol. Comp 2002;6(1):58-73.
8. Van den Bergh F. An analysis of particle swarm optimizers. Ph. D. Thesis, Department of Computer Science, University of Pretoria, Pretoria, South Africa, 2002.
9. Brandstatter B, Baumgartner U, Particle swarm optimization-mass spring system analogy. IEEE Trans. On. Magnetics 2002;38(2):997-1000.
10. Trelea IC, The particle swarm optimization algorithm: convergence analysis and parameter selection. Information Processing Letters 2003;85(6):317-25.
11. Liang JJ, Suganthan PN. Dynamic multi-swarm particle swarm optimiser. Proc. IEEE Int. Symp. on Swarm Intelligence, 2005: 124-129.
12. Yasuda K, Iwasaki N. Adaptive particle swarm optimization. Proc. IEEE Int. Conf. on Systems, Man, and Cybernetics, 2003: 1554-9.
13. Blackwell TM, Particle swarms and population diversity. Soft Computation 2005;9(11):793-802.
14. Blackwell TM. Particle swarms and population diversity I: Analysis. Proc. IEEE Int. Conf. on Genetic and Evolutionary Computation, 2003: 103-7.
15. Campana EF, Fasano G, Pinto A. Dynamic system analysis and initial particles position in particle swarm optimization. Proc. IEEE Int. Symp. on Swarm Intelligence, 2006.
16. Campana EF, Peri G, Pinto A. Particle swarm optimization: Efficient globally convergent modifications. Proc. IEEE Int Conf. on Computational Mechanics, Solids, Structures and Coupled Problems in Engineering, 2006.
17. Zhao H, Mao B. An investigation to Particle trajectories of PSO. Proc. IEEE Int Conf. On Information Engineering and Computer Science, 2009:1-3.
18. Samal NR, Konar A, Das S, Abraham A. A closed-loop stability analysis and parameter selection of the particle swarm dynamics for faster convergence. Proc. IEEE Int Congr. On Evolutionary Computation, 2007:1769-76.
19. Bratton D, Blackwell T. Understanding particle swarms through simplification: a study of recombinant PSO. Proc. ACM Conf. On Genetic and Evolutionary Computation, 2007:2621-8.
20. Clerc M. Stagnation analysis in particle swarm optimization or what happens when nothing happens. Technical Report, Department of Computer Science, University of Essex, 2006.

21. Kadirkamanathan V, Selvarajah K, Fleming PJ. Stability analysis of the particle dynamics in particle swarm optimizer. IEEE Trans. On. Evolutionary Computation 2006;10(3):245-55.
22. Poli R, Langdon WB, Clerc M, Stephens CR. Continuous optimization theory made easy? Finite-element models of evolutionary strategies, genetic algorithms and particle swarm optimizers. Technical Report, Department of Computer Science, University of Essex, 2007.
23. Poli R, Broomhead D. Exact analysis of the sampling distribution for the canonical particle swarm optimiser and its convergence during stagnation. Proc. ACM Conf. On Genetic and Evolutionary Computation, 2007:134-41.
24. Poli R. On the moments of the sampling distribution of particle swarm optimisers. Proc. ACM Conf. On Genetic and Evolutionary Computation, 2007:2907-14.
25. Zhang W, Li H, Zhao Q, Wang H. Guidelines for parameter selection in particle swarm optimisation according to control theory. Proc. IEEE Conf. On Natural Computation, 2009:520-4.
26. Helwig S, Wanka R. Particle swarm optimisation in high-dimensional bounded search spaces. Proc. IEEE Symp. On Swarm Intelligence, 2007:198-205.
27. Chen YP, Jiang. P. Analysis of particle interaction in particle swarm optimisation. Theoretical Computer Science 2010;411:2101-15.
28. Ghosh S, Swagatam D, Debarati K, Kaushik S, Abraham. A. Inter-particle communication and search-dynamics of lbest particle swarm optimizers: An analysis. Information Sciences 2012;182:156-68.
29. Pena J. Theoretical and empirical study of particle swarms with additive stochasticity and different recombination operators. Proc ACM Conf. On Genetic and Evolutionary Computation, 2008:95-102.
30. Helwig S, Wanka R. Theoretical analysis of initial particle swarm behavior. Lecture Notes in Computer Science 2008;5199:889-98.
31. Veeramachaneni K, Osadciw L, Kamath G. Probabilistically driven particle swarms for optimization of multi-valued discrete problems: Design and Analysis. Proc. IEEE Symp. Swarm Intelligence, 2007:141-9.

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