# Ramanujan-Nagell Equation: A Simple Solution

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**Abstract:** In this article we solved the Ramanujan-Nagell equation by using elementary algebraic methods and employing the Lucas- Lehmer numbers. The main objective of this article is to present a simple solution to the equation without going deep into the theorems used in solving the equation. We then assume our solution to be useful for those who are not well familiar with the kind of mathematics that usually imposed in solving the equation. Certainly many different solutions were provided in literature by utilizing different mathematical techniques, some of them are very specialized.
[Omer I. Eid. **Ramanujan-Nagell Equation: A Simple Solution.** *J Am Sci* 2015;11(7):160-162]. (ISSN: 1545-1003). http://www.jofamericanscience.org. 20

**Keywords:** Diophantine equations, Ramanujan-Nagell, Number theory, Algebra, Ramanujan's Numbers.

## 1. Introduction

The equation is first proposed by the genius Indian Srinivasa Ramanujan in 1913 when he asked if there any positive solutions to

$$2^n - 7 = x^2 \tag{1}$$

Other than (n, x) = (3, 1), (4, 3), (5, 5), (7, 11) and (15, 181).[1].

One assumes that the way Ramanujan suggested the question gives us the impression that he had already solved the equation and hence obtained the correct answer, however, there is no reference to confirm this assumption. The equation is considered as one of the famous Diophantine second order equation. It has been tackled by many authors; different approaches in solving the equation have been given in the literature. T. Nagell in 1948 gave nice solution using elementary methods. A translation for his work had appeared in an article published in 1960 [2], and hence the name "*Ramanujan-Nagell equation*". Cohen in his work dedicated to the memory of Trygve Nagell has provided comprehensive references of the history of this equation; he also added good references for the generalized form of the equation [3]

The equation was solved by T. Skolem *etal* [4], they obtained the rational integral solution for n and x as n = 1, 2, 3, 5, and 13. In addition they showed that these are the only solution to the equation $2^{n+2} - 7 = x^2$. In solving the equation they used the *p – adic* method of Skolem. C. Bright [5] showed that the equation can be solved computationally by using simple congruence techniques. S. De. Cheme had introduced the problem as a theorem and gave a detailed proof via the quadratic field $K(\sqrt{-7})$, [6].

## 2. Our Solution

First we note that a key to the solution of the equation will be achieved if one focuses on the term $2^n$. Fortunately, the integer 2 can be written in terms of $\sqrt{\ 7}$ as:

$$2 = \left(\frac{1+\sqrt{-7}}{2}\right)\left(\frac{1-\sqrt{-7}}{2}\right) \tag{2}$$

Note that, in the language of quadratic field $K(\sqrt{-7})$, the above process known as the *unique factorization* of 2.

For further investigation let:

$$a = \left(\frac{1+\sqrt{-7}}{2}\right), \quad b = \left(\frac{1-\sqrt{-7}}{2}\right) \tag{3}$$

It is so easy to verify that both a and b satisfy the quadratic polynomial equation:

$$p_1(y) = y^2 - y + 2 \tag{4}$$

The question now becomes if (3) are the solutions of equation (4), then what will be the polynomial equation that gives the solutions $a^m$ and $b^m$? to answer this question we will next consider the square of a and b

$$a^2 = \left(\frac{-3+\sqrt{-7}}{2}\right), \quad b^2 = \left(\frac{-3-\sqrt{-7}}{2}\right) \tag{5}$$

Which are both the solution to the equation:

$$p_2(y) = y^2 + 3y + 2^2 \qquad (6)$$

And similarly:

$$a^3 = \left(\frac{-5-\sqrt{-7}}{2}\right), \quad b^3 = \left(\frac{-5+\sqrt{-7}}{2}\right) \qquad (7)$$

Are the solutions to:

$$p_3(y) = y^2 + 5y + 2^3 \qquad (8)$$

Now, we step further and obtain a general form, the $m^{th}$ polynomial, starting from $a^3$ and $b^3$. This will also allow us to obtain an expression for the $m^{th}$ solution of a and b. One can easily generate:

$$p_m(y) = y^2 - \alpha(m) + 2^m \qquad (9)$$

Where m is an integer and $\alpha(m)$ satisfies the following sequence, Table (1):

| m | α(m) |
|---|---|
| 1 | **1** |
| 2 | **-3** |
| 3 | **-5** |
| 4 | 1 |
| 5 | **11** |
| 6 | 9 |
| 7 | -13 |
| 8 | -31 |
| ….. | …. |

The above is a well recognized sequence and listed at the *Online Encyclopedia of Integers Sequences* (oeis) with reference number A002249 and formula [7]:

$$\alpha(m) = \alpha(m-1) - 2\,\alpha(m-2) \qquad (10)$$

Next, we will investigate the roots of equation (9) by using standard methods. The quadratic formula for the roots of the general quadratic equation (9) is given by:

$$y_{(1,2)m} = \frac{\alpha(m) \pm \sqrt{\alpha^2(m) - 2^{m+2}}}{2} \qquad (11)$$

Where the corresponding solutions $a^m$ and $b^m$ in terms of $\alpha(m)$ are also generated to have the following form:

$$a^m = \frac{\alpha(m) + \beta(m)\sqrt{-7}}{2} \qquad (12)$$

$$b^m = \frac{\alpha(m) - \beta(m)\sqrt{-7}}{2} \qquad (13)$$

Where $\beta(m)$ is the *Lucas-Lehmer numbers with parameters* $(1+\sqrt{-7})/2$. It is very well known sequence to mathematicians, referenced as A107920, at the oeis [8]. The first few terms are given in Table (2) below:

| m | β(m) |
|---|---|
| 1 | **1** |
| 2 | **1** |
| 3 | **-1** |
| 4 | -3 |
| 5 | **-1** |
| 6 | 5 |
| 7 | 7 |
| 8 | -3 |
| ….. | …. |

Now, on studying equations (11) and (12) one concludes that:

$$\frac{\alpha(m) \pm \sqrt{\alpha^2(m) - 2^{m+2}}}{2} = \frac{\alpha(m) + \beta(m)\sqrt{-7}}{2}$$

This gives:

$$\sqrt{\alpha^2(m) - 2^{m+2}} = \beta(m)\sqrt{-7} \qquad (14)$$

By squaring both sides and rearranging terms one finds that:

$$2^{m+2} - 7\beta^2(m) = \alpha^2(m) \qquad (15)$$

Which can be transferred to Ramanujan-Nagell equation *if and only if* $\beta^2(m) = 1$, or

$$\beta(m) = \pm 1 \qquad (16)$$

Of course x is corresponding to $\alpha(m)$ and n goes to $(m+2)$.

J. Cassels [9] has proved the condition given by (16) by using the technique of *power series expansion* of $\beta(m)$ together with the application of *Strassmann's theorem*. It is now left to obtain the values of m that satisfy equation (16). One sees from the sequence $\beta(m)$ that the only values of m satisfying equation (16) are:

$$m = 1, 2, 3, 5, \text{ and } 13 \qquad (17)$$

Consider for example m = 1, this gives n = 3 and x = 1, which is fully gives Ramanujan numbers for (n, x) = (3, 1). Similarly for the other values of m. This result implies that the solution to the Ramanujan-Nagell equation is obtained only when m takes the values specified by (16).

**3. Advantages of Our Solution**

1. The given solution for the Ramanujan-Nagell equation is simple, straight forward in which we used elementary algebraic methods, and we make use of the integers listed at the oeis website.
2. It gives the Ramanujan numbers of x and n directly without the need to specify odd and even cases separately.
3. In our solution we did not need to go through *the quadratic field theory* as others authors did in solving the equation. However, one of its results is emerged automatically and is used indirectly in our results.
4. The analysis of the problem implied that some constrained had to be imposed, we proposed a constrained on the series β(m) to have the values of ±1. Fortunately the same constraint on the series "the group of units of the quadratic field $Q\sqrt{-7}$ is {±1}" had been concluded by S. De Chenne [6] and proved in details by J. Cassels [9].
5. The results obtained are in accordance with the proof given previously by T Skolem *etal* [3] who showed that there are no solutions for the Ramanujan-Nagell equation past $2^{15}$, this is equivalent to the result obtained in our study that the series $\beta(m) \neq \pm1$ when m > 13. (i.e. n>15).
6. The polynomial obtained in our work is actually the characteristic equation of the recurrence relation for the Lucas sequences with P and Q being α(m) and $2^m$ respectively.
7. It is worth to mention that the condition imposed on $\beta^2(m) = 1$ implies that both positive and negative values for x are given.

6/16/2015

**References**


1. S. Ramanujan, Question 464, *J of the Indian Mathematical Society* **5** (1913). 120.
2. T. Nagell, "The Diophantine Equation $x^2 + 7 = 2^n$", *Arkiv för Mathematik* Band **4** nr 13, (1960), 185-187.
3. E. L Cohen, "On the Ramanujan-Nagell Equation and its Generalizations", Number Theory (Banff AB, 1988), de Gruyter, Berlin, (1990), 81-92.
4. T. Skolem, S. Chowla and D. J. Lewis, "The Diophantine Equation $2^{n+2} - 7 = x^2$ and Related Problems", *Proc. Amer. Math. Soc.* **10** (1959), 663-669.
5. C. Bright, "Solving the Ramanujan's Square Equation Computationally", https://cs.uwaterloo.ca/~cbright/nsra/ramanujans-square-equation.pdf, (2007), 1-4.
6. S. De Chenne, "The Ramanujan-Nagell Theorem: Understanding the Proof", http://buzzard.ups.edu/courses/2013spring/projects/ spencer-ant-ups-434-2013.pdf, (2013), 1-9.
7. https://oeis.org/A002249.
8. https://oeis.org/A107920.
9. J. W. S. Cassels, "Local Fields", Cambridge Univ. Press. Cambridge, (1986).