Multipoint Iterative Method with Cubic Convergence

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Abstract: Finding the zeros of nonlinear functions has wide applications in engineering, in scientific computations and various other fields. In these problems we seek methods that lead to approximate solutions. Sometimes the applications of the iterative methods depending on the derivatives are restricted in engineering. In this paper, we propose some iterative formulae without derivatives. The convergence analysis shows that these methods are cubically convergent. The best property of these schemes is that they are derivative free. Several numerical examples are given to illustrate the efficiency and performance of the proposed methods.
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1.1 Introduction

In recent years, much attention has been given to develop several iterative methods for solving nonlinear equations. These methods can be classified as one-step and two-step methods. Two-step methods have been suggested by combining the well-known Newton method with other one-step methods, see [5–8] and the references therein. Abbasbandy [1] and Chun [3] has proposed and studied several one-step and two-step iterative methods with higher order convergence by using the decomposition technique of Adomian [2]. In their methods, they have to used the higher order differential derivatives which is a serious drawback.

To overcome this draw back, we suggest and analyze a family of multiple-step methods for solving nonlinear equations using a different type of decomposition, which does not involve the high-order differentials of the function. This decomposition method is essentially due to Daftardar-Gejji and Jafari [4]. Our method is very simple as compared with Adomian decomposition method. In particular, we introduce a new three-step iterative method for solving nonlinear equations. We also prove convergence of the proposed method. Several numerical examples are given to illustrate the efficiency and the performance of the new ideas.

We discuss the local convergence of a derivative-free eighth order method in a Banach space setting. The present study provides the radius of convergence and bounds on errors under the hypothesis based on the first Fréchet-derivative only. The approaches of using Taylor expansions, containing higher order derivatives, do not provide such estimates since the derivatives may be nonexistent or costly to compute. By using only first derivative, the method can be applied to a wider class of functions and hence its applications are expanded. Numerical experiments show that the present results are applicable to the cases wherein previous results cannot be applied.

A class of third order convergent methods proposed by Helley [1994] requires the evaluation of second order derivative of $f(x)$, which is practically difficult in majority of cases. New multipoint iterative methods for non linear equations have been proposed by Ortega and Rheinboldt [1970], Potra and Ptak [1984], Frontini and Sormani [2003 a, b], Weerakoon and Fernando [2000], Ozban [2004] and Homeier [2003, 2005]. These techniques fail, if the derivative of the function is zero or very small near the root. The authors have proposed a linearly convergent square root type method for solving scalar non-linear equations.

1.2 Circle Technique for Third Order Formula

Let $f(x)=0,$ be an equation whose solution is required. Suppose its graph is represented by $y=f(x)$. Take x_0 as initial approximation and $x_1=x_0+h$ as better approximation of the root, where h being a small quantity. Draw a circle of radius $f(x_0)$ centre $[x_0, f(x_0)]$ on $y=f(x)$. The equation of circle is given by,

$$(x-x_0)^2+[y-f(x_0)]^2=[f(x_0)]^2 \qquad \text{…(1.1)}$$

If the circle passes through the point $[x_1, f(x_0+h)]$, then equation (1.1) takes the form,

$$h^2 + [f(x_0+h)]^2 - 2f(x_0)f(x_0+h) = 0. \qquad \ldots(1.2)$$

Expanding by Taylor's theorem and simplifying, one gets,

$$h = \frac{f(x_0)}{\sqrt{1+f'^2(x_0)}} \qquad \ldots(1.3)$$

The successive approximations are taken until the circle becomes a point on the x-axis, the general formula can be given by,

$$x_{n+1} = x_n \pm \frac{f(x_n)}{\sqrt{1+f'^2(x_n)}}, \quad n \ge 0. \qquad \ldots(1.4)$$

This formula has guaranteed convergence unlike Newton's method but its order of convergence is found to be one. Another iterative technique has been proposed by the authors, similar to the one given in Chapter 6 by equation (6.4). In this equation the value of $u(x_n)$ is taken equal to the value of *h* given by equation (1.3), therefore,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'[x_n \pm k\, u(x_n)]}, \quad n \ge 0, \qquad \ldots(1.5)$$

where $u(x_n) = \frac{f(x_n)}{\sqrt{1+f'^2(x_n)}}$ …(1.6)

and k is chosen as a free parameter. It require one evaluation of function and two evaluations of its derivatives for every non-zero value of the parameter k.

1.3 Convergence of Method

Let α is a root of $f(x)=0$. An approximation of the root is given by $x_n = \alpha + e_n$, where e_n is error. Taylor's series expansions given by equations (2.9) and (2.10) of $f(x_n)$ and $f'(x_n)$ are used.

Case I : When $|f'(x_n)| \le 1,$ then $\sqrt{1+f'^2(x_n)} \cong 1$ and $|f'(\alpha)| \cong 1$, after some simplification equation (1.6) gives,

$$u(x_n) = \frac{f(x_n)}{\sqrt{1+f'^2(x_n)}} = f'(\alpha)\left[e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\right],$$

and

$$f'[x_n \pm k\,u(x_n)] = f'(\alpha)\left\{1 + 2C_2[1 \pm k\, f'(\alpha)]e_n \pm 2kC_2^2 f'(\alpha)e_n^2 + 3C_3[1 \pm k\, f'(\alpha)]^2 e_n^2 + O(e_n^3)\right\} \qquad \ldots(1.7)$$

Using equation (1.7) in equation (1.5) and expanding by using $|f'(\alpha)| \cong 1$, one gets,

$$e_{n+1} = C_2(1 \pm 2k)e_n^2 + \left[2C_2^2(1 \pm k) \pm 2k\,C_2^2 - 4C_2^2(1 \pm k)^2 + 3C_3(1 \pm k)^2 - C_3\right]e_n^3 + O(e_n^4) \qquad \ldots(1.8)$$

To have third-order convergence, the coefficient of e_n^2 must vanish, which gives,

$$1 \pm 2k = 0 \text{ or } k = \mp 1/2. \qquad \ldots(1.9)$$

Case II : When $|f'(x_n)| > 1,$ then $\sqrt{1+f'^2(x_n)} \cong f'(x_n)$, it gives the same error equation and value of k is thus obtained as above in Case I. Therefore, for $k = \mp 1/2$, iterative method equation (1.5) becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left[x_n \mp \frac{f(x_n)}{2\sqrt{1+f'^2(x_n)}}\right]}. \qquad \ldots(1.10)$$

where the sign in the denominator should be chosen such that it becomes largest in magnitude.

1.4 Numerical Examples

The results of numerical examples given in Table 6.1 in Chapter 6, are found with the help of equation (1.10). The efficiency and accuracy of the Newton's method, Weerakoon & Fernando [2000] and the method developed in equation (1.10) have been compared by the authors. The termination criterion was taken as $\left|f(x_n)\right| < 1.0 \times 10^{-15}$.

1.5 Conclusions

It has been observed that the method proposed in equation (1.10) has cubic convergence. This method does not fail even when $f'(x) = 0$ in the neighbourhood of the required root. Therefore, this technique can be used as an alternative to existing techniques or in cases where existing techniques fail and has well known geometric derivation.

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