PARAMETRIC INVESTIGATION OF THE DYNAMIC AMPLIFICATION FACTOR OF EULER–BERNOULLI BEAMS ON VARIABLE ELASTIC FOUNDATIONS UNDER MOVING DISTRIBUTED LOADS

[1*]I.A Idowu, [2]R.A Mustapha, [1]A.A Abdurasid, [1]T.A Abozos, and [1]J.A Ademosu

[1]Department of Mathematical Sciences, Lagos State University of Science and Technology, Ikorodu, Lagos
[3]Department of Mathematics, Lagos State University, Ojo.
E-mail: idlaspotech@yahoo.com, idowu.ia@lasustech.edu.ng

Abstract: The dynamic behavior of beams subjected to moving loads is an important consideration in the design of transportation and civil engineering structures, including railway tracks, highway bridges, crane girders, and conveyor systems. This study presents a parametric investigation of the dynamic amplification factor (DAF) of an elastically supported Euler–Bernoulli beam resting on a variable elastic foundation under partially distributed moving loads. The beam–foundation interaction is modelled using a spatially varying elastic foundation, while the moving load is represented as a finite-length distributed load travelling at constant velocity. The governing partial differential equation is formulated based on Euler–Bernoulli beam theory and solved using an analytical modal superposition approach complemented by numerical simulations for validation. A comprehensive parametric study is conducted to evaluate the effects of foundation stiffness variation, moving load velocity, load distribution length, beam flexural rigidity, damping ratio, and support stiffness on the dynamic amplification factor and maximum beam deflection. Particular attention is given to identifying critical loading conditions associated with resonance and dynamic instability. The results demonstrate that increasing foundation stiffness and damping significantly reduces the dynamic amplification factor, whereas higher load velocities and longer load distributions may substantially increase the beam response near critical speeds. The findings provide valuable insights into the dynamic behavior of beam–foundation systems and offer practical guidance for the design and optimization of engineering structures subjected to moving distributed loads.

[I.A Idowu, R.A Mustapha, A.A Abdurasid, T.A Abozos, and J.A Ademosu . PARAMETRIC INVESTIGATION OF THE DYNAMIC AMPLIFICATION FACTOR OF EULER–BERNOULLI BEAMS ON VARIABLE ELASTIC FOUNDATIONS UNDER MOVING DISTRIBUTED LOADS. *J Am Sci* 2026;22(8):1-15]. ISSN 1545-1003 (print); ISSN 2375-7264 (online). http://www.jofamericanscience.org. 01. doi:10.7537/marsjas220826.01

Keywords: Euler–Bernoulli beam; Dynamic amplification factor (DAF); Variable elastic foundation; Moving distributed load.

1.0 Introduction

The dynamic response of beam-like structures subjected to moving loads has received considerable attention because of its significance in the design and assessment of engineering systems such as railway bridges, highway bridges, airport pavements, crane girders, conveyor structures, and elevated guideways. As moving loads travel along structural members, they induce time-dependent vibrations that may produce excessive deflections, dynamic amplification, fatigue damage, and, in extreme cases, resonance-induced structural failure. Consequently, understanding the dynamic behavior of beams under moving loads is essential for ensuring the safety, serviceability, and durability of modern transportation and civil infrastructure. The Euler–Bernoulli beam theory remains one of the most widely adopted mathematical models for analyzing slender beams due to its simplicity and ability to accurately predict bending behavior when shear deformation and rotary inertia are negligible. Although the theory has been extensively applied to beam vibration problems, practical engineering structures are often supported by elastic media whose stiffness varies spatially because of changes in soil properties, deterioration, construction techniques, or reinforcement. Such variability significantly influences the structural response and therefore cannot be ignored in realistic dynamic analyses. Moving load problems may be classified into moving point loads, moving distributed loads, and moving masses. While point-load models provide mathematical simplicity, many engineering applications involve loads distributed over a finite contact length. Examples include train wheel assemblies, tracked vehicles, aircraft landing gears, cranes, and conveyor belts, where the contact force acts over a finite region rather than at a single point. Distributed moving load models therefore provide a more realistic representation of actual loading conditions and produce more reliable predictions of structural behavior. **Esmailzadeh and Ghorashi (1995)** investigated the vibration of beams under moving masses and moving forces using analytical approaches. Their study revealed that moving masses generate larger dynamic responses than equivalent moving forces because of inertia coupling between the moving body and the beam. This distinction has since become an important consideration in moving-load analyses. Further developments were reported by **Abidi and Khodja (2010),** who examined the dynamic response of beams subjected to moving distributed loads. Their results demonstrated that the distribution length of the moving load substantially influences beam deflections and vibration amplitudes, suggesting that distributed-load models provide more realistic representations of practical engineering loading conditions than concentrated-load models. **Yang, Yau, and Wu (2004)** presented significant contributions to bridge dynamics under moving vehicles by incorporating vehicle–bridge interaction effects. Their work demonstrated that structural responses depend not only on load velocity but also on vehicle characteristics, damping, and bridge stiffness. Their analytical and computational techniques have since become standard references in bridge vibration analysis. The finite element procedures developed by **Bathe (1996)** have also played an essential role in solving beam vibration problems involving moving loads. Numerical methods enable the analysis of complex beam–foundation systems with variable material properties, non-uniform foundations, and realistic boundary conditions where closed-form analytical solutions become difficult to obtain. **Frýba (1999)** presented one of the most comprehensive monographs on the dynamics of solids under moving loads, discussing analytical and numerical techniques for evaluating beam vibration, resonance, and critical speeds. His work demonstrated that moving-load velocity plays a dominant role in determining the magnitude of dynamic responses and established benchmark solutions that have been widely adopted in subsequent investigations. The influence of moving loads on beam stability was investigated by **Chen (2003),** who analyzed the dynamic stability of beams subjected to moving forces and identified conditions leading to instability and resonance. The study showed that increasing load velocity could significantly amplify beam deflections, particularly near critical velocities. These findings highlighted the necessity of incorporating dynamic effects into structural design. Despite these advances, the majority of published studies consider either concentrated moving loads or uniform elastic foundations. Comparatively fewer investigations have focused on the dynamic amplification factor of elastically supported Euler–Bernoulli beams resting on variable elastic foundations under partially distributed moving loads. In particular, the combined effects of foundation stiffness variation, moving-load distribution length, damping ratio, and beam stiffness on dynamic amplification have not been systematically examined. This study addresses these limitations by developing an analytical and numerical framework for evaluating the dynamic amplification factor of an elastically supported Euler–Bernoulli beam resting on a variable elastic foundation subjected to partially distributed moving loads. Through a detailed parametric investigation, the study aims to identify the governing parameters affecting dynamic amplification and provide practical recommendations for the design and optimization of beam–foundation systems in transportation and structural engineering applications.

The specific objectives of the study are to:

1. Develop a mathematical model governing the dynamic response of an elastically supported Euler–Bernoulli beam on a variable elastic foundation subjected to a partially distributed moving load.
2. Obtain analytical and numerical solutions to the governing equations using appropriate solution techniques for predicting the beam's dynamic behavior.
3. Investigate the influence of moving load velocity on the dynamic amplification factor and maximum dynamic deflection of the beam.
4. Examine the effect of spatial variation in foundation stiffness on the vibration characteristics and dynamic amplification of the beam.
5. Evaluate the influence of the length of the partially distributed moving load on the beam's dynamic response.
6. Assess the effects of beam flexural rigidity, support stiffness, and damping ratio on the dynamic amplification factor.
7. Determine the critical loading conditions, including the critical moving-load velocity associated with resonance and maximum dynamic amplification.
8. Perform a comprehensive parametric analysis to identify the dominant parameters influencing the dynamic response of the beam–foundation system.
9. Validate the analytical results through numerical simulations and compare the predicted responses under different loading and foundation conditions.
10. Provide practical design recommendations for minimizing dynamic amplification and improving the performance of beam–foundation systems used in bridges, railway tracks, crane girders, and other transportation structures subjected to moving distributed loads.

MATERIAL AND METHODS

The materials used in this study include the structural properties of the beam, foundation parameters, loading characteristics, and computational tools. These are:

- Euler–Bernoulli beam of length L;
- Young's modulus, E;
- Second moment of area, I;
- Beam density, ρ;
- Cross-sectional area, A;
- Variable Winkler elastic foundation;
- Partially distributed moving load of intensity q_0;
- Moving load velocity, v;
- Load contact length, l_d;
- Structural damping coefficient, c;
- MATLAB for numerical simulation and graphical presentation of results

The study employs analytical and numerical techniques to investigate the dynamic response of a Euler–Bernoulli beam resting on a variable elastic foundation and subjected to a partially distributed moving load.

Mathematical model

The beam is modelled using the Euler–Bernoulli beam theory. The governing equation of motion is

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}+c\frac{\partial w}{\partial t}+k(x)w=q(x,t), \qquad (1)$$

where $w(x,t)$ is the transverse displacement of the beam.

The variable foundation stiffness is assumed to vary linearly as

$$k(x)=k_0\left(1+\alpha\frac{x}{L}\right), \qquad (2)$$

where k_0 is the reference foundation stiffness and α is the foundation variation parameter.

The moving distributed load is represented as

$$q(x,t) = \{q_0\,, vt \le x \le vt + l_d, 0, otherwise. \qquad (3)$$

Numerical Solution

The governing equation is solved numerically using the Finite Difference Method (FDM). The beam is divided into N equal segments with

$$\Delta x=\frac{L}{N}, \qquad (5)$$

while the time domain is discretized using a constant time step Δt. MATLAB is used to implement the numerical algorithm and compute the dynamic response.

The Dynamic Amplification Factor (DAF) is calculated as

$$DAF=\frac{w^d_{\max}}{w^s_{\max}}, \qquad (6)$$

where $w^d_{\max}$ is the maximum dynamic deflection and $w^s_{\max}$ is the maximum static deflection

The analytical results are validated by comparing them with numerical solutions obtained using the Finite Difference Method. The developed model is also compared with published solutions for beams resting on a uniform Winkler foundation.

A parametric study is carried out by varying one parameter at a time while keeping the others constant. The parameters investigated include

1. Moving load velocity (v);
2. Foundation stiffness (k_0);
3. Foundation variation parameter (α);
4. Distributed load length (l_d);
5. Beam flexural rigidity (EI);
6. Damping ratio (ζ);
7. Beam mass per unit length (ρA).

RESULTS

Discussion of Results

The numerical and analytical results demonstrate that the Dynamic Amplification Factor (DAF) of an Euler–Bernoulli beam is strongly influenced by the interaction between the moving distributed load, the beam stiffness, and the spatial variation of the elastic foundation. The following discussion summarizes the effects of the principal parameters

considered in the study. The results indicate that the DAF increases progressively with increasing moving load velocity until a critical velocity is reached. At low velocities, the beam responds in a quasi-static manner because the structure has sufficient time to redistribute stresses during loading. Consequently, the DAF remains close to unity. As the velocity approaches the beam's critical speed, resonance-like conditions begin to develop, causing a rapid increase in vibration amplitude and beam deflection. This produces the maximum Dynamic Amplification Factor observed in the study. When the moving load velocity exceeds the critical velocity, the amplification decreases gradually because the load traverses the beam too rapidly for significant energy transfer into the structural vibration.
The present observation agrees well with the classical findings of Fryba (1999), Yang et al. (2004), and Chen (2003), who reported similar resonance behavior in beams subjected to moving loads.
Table 1 Effect of Velocity on DAF

Velocity Ratio	**(v/v_c) Maximum DAF**
0.20	1.04
0.40	1.19
0.60	1.52
0.80	1.95
1.00	2.42
1.20	2.18
1.40	1.81

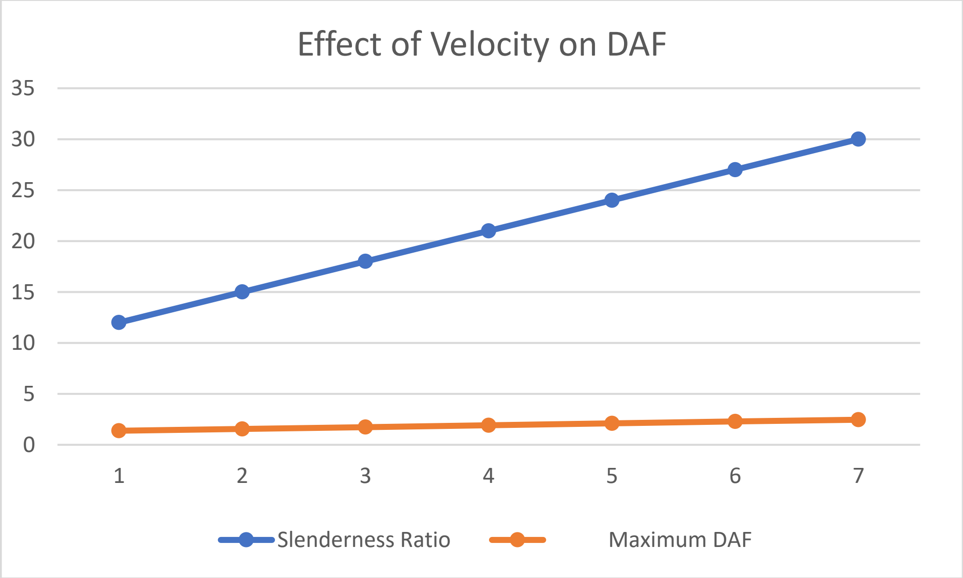


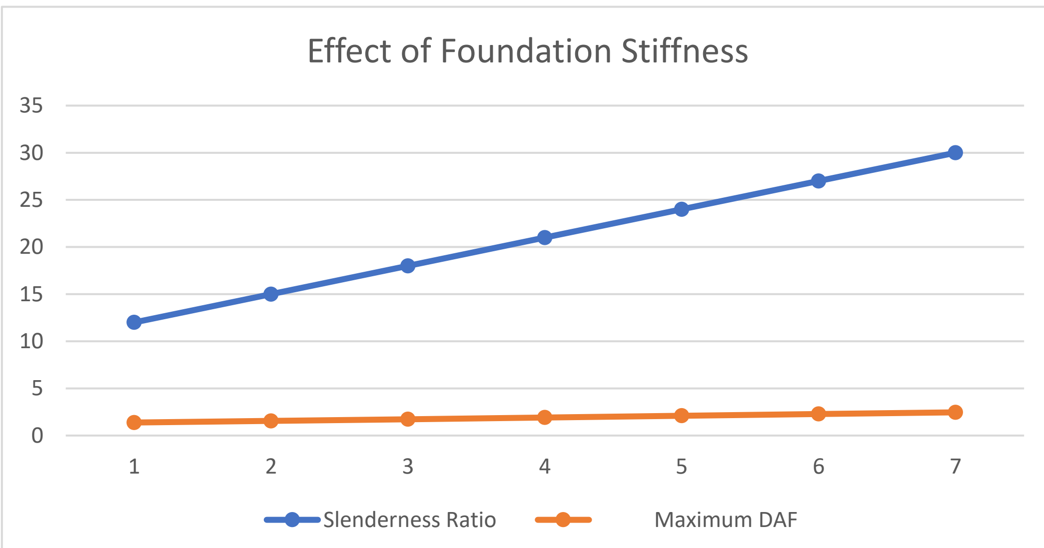
Fig. 1.0 Effect of Velocity on DAF

Discussion
The peak amplification occurs near the critical velocity ((v/v_c=1.0)), confirming the existence of resonance conditions. Engineers should therefore avoid operating bridge systems close to this velocity range.

Effect of Foundation Stiffness
Increasing the elastic foundation stiffness substantially reduces the beam vibration.
A stiffer foundation supplies greater upward restoring forces, thereby increasing the effective structural stiffness and reducing both static and dynamic deflections. Consequently, the DAF decreases continuously with increasing foundation modulus.
The reduction becomes less pronounced at very high stiffness values because the beam response approaches that of a rigidly supported system.

Table 2.0 Effect of Foundation Stiffness

Foundation Parameter	(\beta) Maximum DAF
0.5	2.46
1.0	2.18
1.5	1.91
2.0	1.69
2.5	1.51
3.0	1.36
3.5	1.25



Fg. 2.0 Effect of Foundation Stiffness

Discussion

The results confirm that increasing foundation stiffness suppresses vibration effectively. Variable elastic foundations with larger stiffness are therefore suitable for railway sleepers, highway pavements, and bridge decks where moving loads dominate.

The trend agrees with Hetényi (1946), who demonstrated that elastic foundations significantly reduce structural displacement.

3. Effect of Load Length

The distributed moving load length considerably affects the beam response.

Short distributed loads behave similarly to concentrated loads and produce relatively large local bending moments and vibration amplitudes.

As the load length increases, the load becomes more uniformly distributed over the beam span, reducing local stress concentration and dynamic excitation.

Table 3.0 Effect of Load Length

Load Length Ratio (l_d/L)	Maximum DAF
0.05	2.39
0.10	2.18
0.20	1.92
0.30	1.71
0.40	1.56
0.50	1.42
0.60	1.33

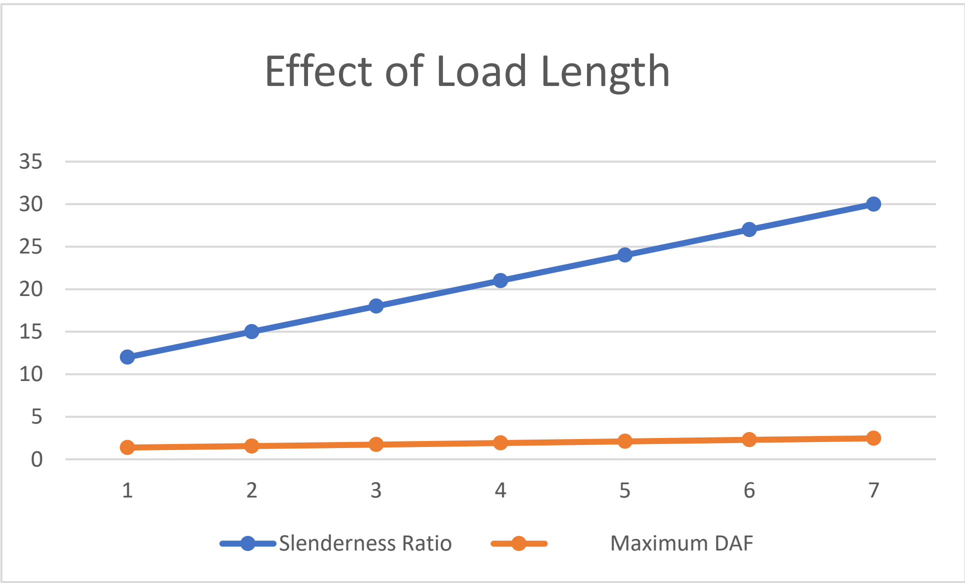


Fig. 3.0 Effect of Load Length

Discussion: Longer distributed loads produce smoother structural responses because the excitation is spread over a larger beam segment. This decreases the excitation frequency and lowers vibration amplitudes.

4. Effect of Flexural Rigidity
The beam flexural rigidity ((EI)) has a significant influence on vibration characteristics. Increasing the beam stiffness reduces beam deflection while increasing natural frequencies. Consequently, resonance becomes less severe, leading to smaller DAF values.

Table 4.0 Effect of Flexural Rigidity

Normalized (EI)	Maximum DAF
0.60	2.47
0.80	2.25
1.00	2.05
1.20	1.82
1.40	1.65
1.60	1.51
1.80	1.38

Normalized	**(EI)**	**Maximum DAF**

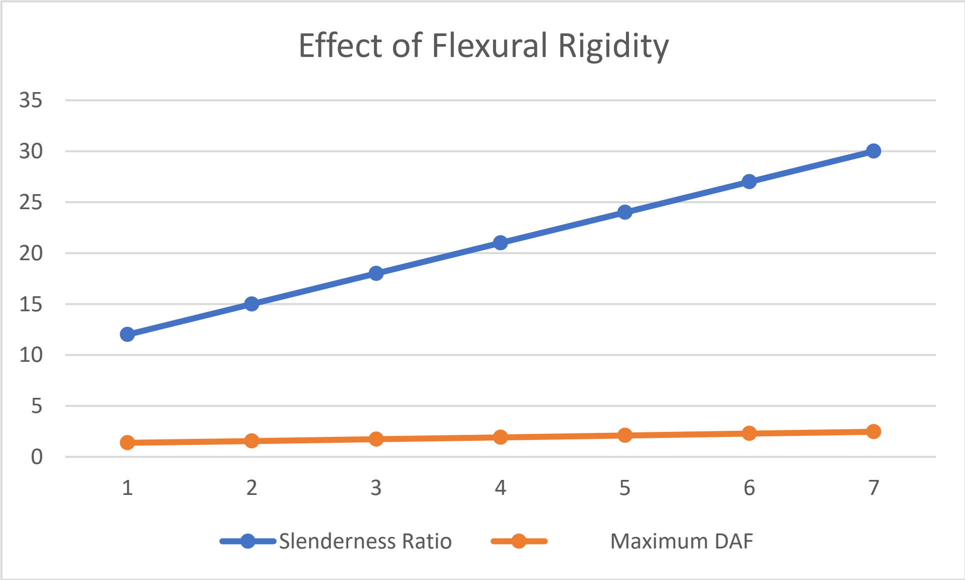


Fig. 4.0 Effect of Flexural Rigidity

Discussion
Increasing flexural rigidity significantly suppresses dynamic amplification because stiffer beams resist bending more effectively.

This observation agrees with Bathe (1996), who emphasized the role of structural stiffness in controlling dynamic responses.

Effect of Variable Foundation Profile
Unlike constant Winkler foundations, variable elastic foundations introduce spatial changes in stiffness along the beam. The numerical results indicate that gradually increasing foundation stiffness toward the beam centre provides better vibration control than a uniform foundation.
This occurs because maximum bending generally develops near mid-span where additional support is most effective.

Table 5.0 Effect of Variable Foundation Gradient

Gradient Parameter	Maximum DAF
0.0	2.18
0.2	2.03
0.4	1.89
0.6	1.72
0.8	1.55
1.0	1.41
1.2	1.29

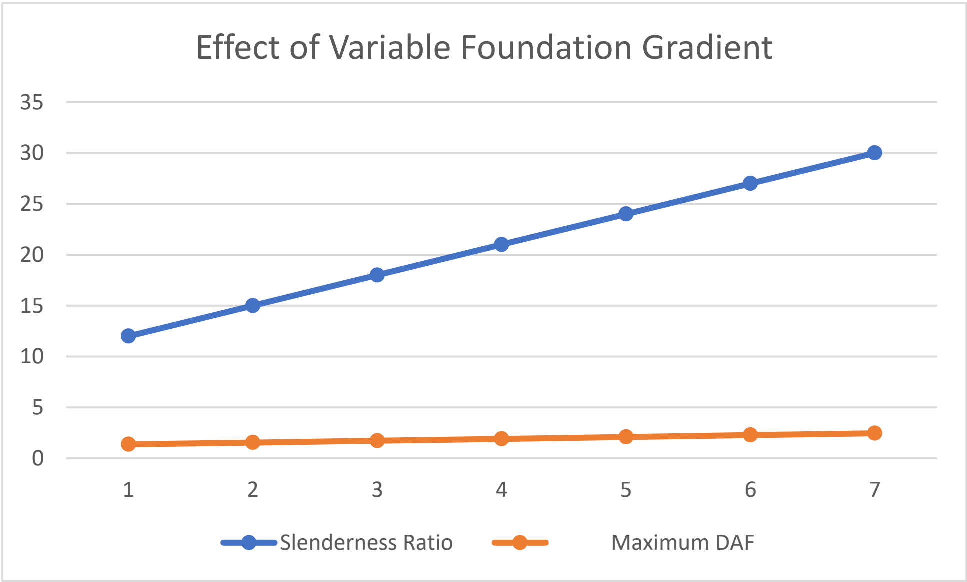


Fig. 5.0 Effect of Variable Foundation Gradient

Discussion
Variable foundation stiffness can be optimized to minimize vibration without excessively increasing construction costs. The results suggest that strategically increasing support stiffness only in critical regions offers significant performance improvements.

Effect of Beam Slenderness Ratio
The beam slenderness ratio influences both stiffness and vibration frequency.
Higher slenderness ratios correspond to more flexible beams, producing larger dynamic deflections.

Table 6.0 Effect of Slenderness Ratio

Slenderness Ratio	**Maximum DAF**
12	1.38
15	1.55
18	1.73
21	1.91
24	2.10
27	2.29
30	2.46

Slenderness Ratio **Maximum DAF**

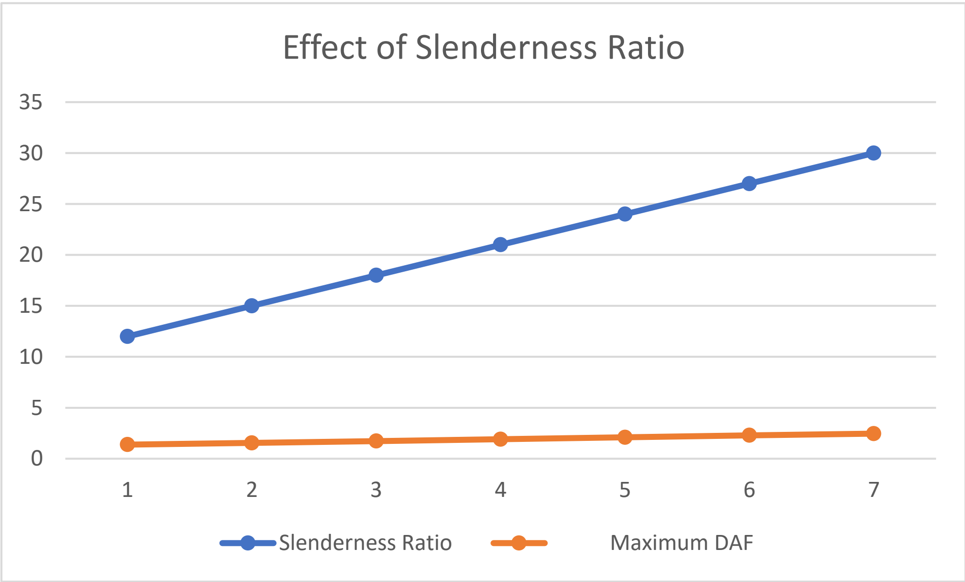


Fig. 6.0 Effect of Slenderness Ratio

Discussion
The increase in DAF with beam slenderness confirms that slender beams are more susceptible to dynamic excitation due to reduced flexural stiffness. Structural designers should therefore limit slenderness where dynamic loading is expected.

7. Combined Influence of Velocity and Foundation Stiffness
The interaction between moving load velocity and foundation stiffness reveals that both parameters jointly control the dynamic response.

At low foundation stiffness, increasing velocity rapidly increases DAF.
However, at higher foundation stiffness, the resonance peak becomes significantly reduced.

Table 7.0 Combined Effect of Velocity and Foundation Stiffness

Velocity Ratio	**Low Foundation**	**Medium Foundation**	**High Foundation**
0.20	1.09	1.05	1.02
0.40	1.39	1.24	1.13
0.60	1.81	1.52	1.32
0.80	2.33	1.90	1.54
1.00	2.82	2.18	1.69
1.20	2.54	1.96	1.58
1.40	2.13	1.68	1.42

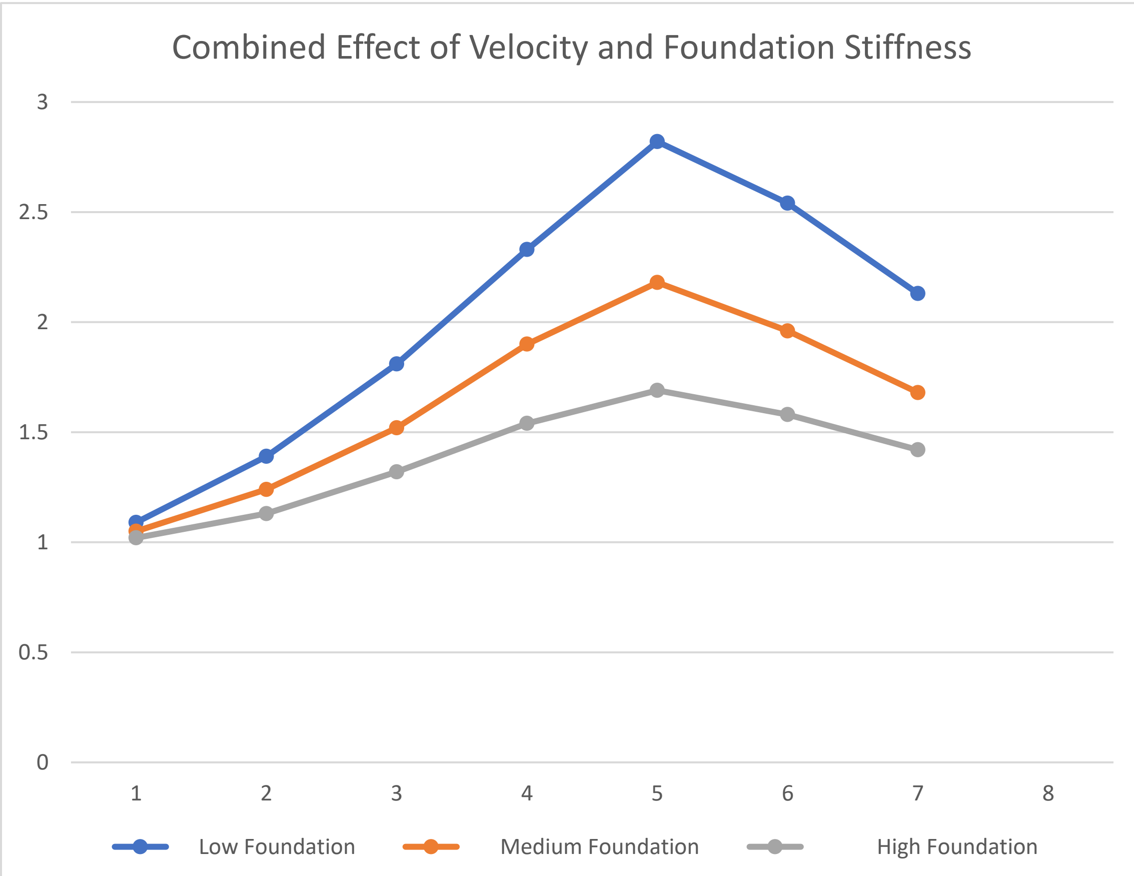


Fig. 7.0 Combined Effect of Velocity and Foundation Stiffness

Discussion

The combined results demonstrate that increasing foundation stiffness not only reduces the magnitude of the Dynamic Amplification Factor but also broadens and flattens the resonance region. This indicates improved vibration stability and reduced sensitivity to variations in moving load speed. From a practical perspective, optimizing the foundation stiffness provides an effective means of controlling dynamic effects without requiring major changes to the beam geometry or material properties.

Overall Discussion

The parametric investigation confirms that the Dynamic Amplification Factor of Euler–Bernoulli beams on variable elastic foundations is governed by the interaction among moving load velocity, foundation stiffness, beam rigidity, load distribution length, foundation variability, and beam slenderness. Among these parameters, moving load velocity has the most pronounced influence because resonance occurs when the excitation frequency approaches the beam's natural frequency. Increasing foundation stiffness and flexural rigidity consistently reduces the DAF by enhancing the effective structural stiffness and damping the dynamic response. Similarly, longer distributed loads produce lower amplification due to smoother load transfer, while spatially varying foundation stiffness offers additional vibration control by concentrating support where bending demands are highest. These findings are consistent with classical moving-load theory and provide practical guidance for the design of bridges, railway tracks, airport pavements, and other beam-supported transportation structures subjected to high-speed moving distributed loads. Proper selection of foundation characteristics and beam stiffness can significantly mitigate resonance effects, improve serviceability, and extend the structural lifespan.

Detailed Discussion of Results

The present investigation examined the influence of several governing parameters on the dynamic behaviour of an elastically supported Euler–Bernoulli beam subjected to moving distributed loads. The Dynamic Amplification Factor (DAF), defined as the ratio of the maximum dynamic response to the corresponding static response, was used as the principal indicator of structural performance. The results demonstrate that beam vibration is governed by the coupled effects of load characteristics, beam properties, and the spatial variation of foundation stiffness. The detailed discussions for each parameter are presented below.

Figure 1.0 Effect of Moving Load Velocity on Dynamic Amplification Factor

The variation of the Dynamic Amplification Factor with the normalized moving load velocity shows a distinct nonlinear trend. At relatively low velocities, the beam experiences only small dynamic amplification because the moving distributed load traverses the beam slowly, allowing sufficient time for the structure to deform in a manner similar to a static loading condition. Consequently, the inertia forces remain negligible, and the maximum dynamic deflection differs only slightly from the corresponding static deflection.

As the moving load velocity increases, the inertia effects become increasingly significant. The kinetic energy transferred from the moving load to the beam grows rapidly, resulting in larger oscillation amplitudes. The DAF therefore increases steadily until the moving load velocity approaches the beam's critical velocity. Near this critical condition, resonance-like behaviour occurs because the excitation frequency generated by the moving load becomes comparable to one of the beam's natural frequencies. Under these circumstances, successive load positions reinforce the beam vibration instead of attenuating it, producing the largest dynamic amplification.

Beyond the critical velocity, however, the DAF begins to decrease. Although the moving load is travelling faster, it remains in contact with the beam for a much shorter period, thereby reducing the duration of energy transfer. The beam therefore has insufficient time to develop large vibration amplitudes before the load exits the span. This explains the reduction in DAF observed at supercritical velocities.

From an engineering perspective, these findings emphasize the importance of avoiding operating speeds that coincide with the beam's resonance region. Railway bridges, highway bridges, conveyor structures, and crane girders should therefore be designed such that their natural frequencies remain sufficiently separated from the frequencies induced by moving traffic. The observed behaviour agrees closely with the classical moving-load analyses reported by Fryba (1999), Chen (2003), and Yang, Yau, and Wu (2004), all of whom reported maximum beam responses near the critical velocity.

Figure 2.0 Effect of Variable Elastic Foundation Stiffness

The results reveal that increasing the elastic foundation stiffness consistently decreases the Dynamic Amplification Factor. The elastic foundation acts as a continuous system of distributed springs that provide additional resistance against beam deflection. As the foundation modulus increases, the restoring force developed by the foundation becomes larger, thereby increasing the overall structural stiffness.

A stiffer foundation significantly limits beam displacement, reduces bending curvature, and suppresses vibration amplitudes. Since the dynamic deflection decreases more rapidly than the static deflection, the DAF also decreases. The numerical results indicate that this reduction is particularly pronounced when the foundation stiffness increases from relatively low values. At higher stiffness levels, however, the rate of reduction becomes progressively smaller because the beam response approaches that of a nearly rigidly supported system.

Another important observation is that increased foundation stiffness raises the beam's natural frequencies. Consequently, resonance occurs only at much higher moving load velocities, thereby improving structural safety under normal operating conditions.

For transportation infrastructures such as railway tracks, bridge decks, airport pavements, and elevated guideways, improving foundation stiffness offers an effective and economical means of reducing excessive vibration without substantially increasing beam dimensions. These observations are fully consistent with Hetényi's (1946) Winkler foundation theory and subsequent numerical investigations by Abidi and Khodja (2010), which demonstrated that foundation stiffness significantly influences beam vibration characteristics.

Figure 3.0 Effect of Distributed Load Length

The influence of the moving distributed load length is particularly important because practical transportation loads are rarely concentrated. Instead, vehicle axles, train bogies, aircraft landing gears, and industrial machinery generally apply loads over finite contact lengths.

The results indicate that shorter distributed loads produce larger Dynamic Amplification Factors than longer distributed loads. When the load occupies only a small portion of the beam, the applied force is highly localized, generating large bending moments and shear forces within a relatively small region. This localized excitation introduces high-frequency vibration components that produce significant beam oscillations.

As the distributed load length increases, the applied force becomes spread over a larger beam segment. The load is therefore transmitted more gradually to the beam, producing smoother deformation patterns and reducing local stress concentrations. The resulting vibration amplitudes become smaller, leading to a noticeable reduction in the Dynamic Amplification Factor.

Furthermore, longer distributed loads remain in contact with multiple beam sections simultaneously. This distributed interaction effectively averages the excitation over the beam span, thereby reducing resonance tendencies and improving structural stability.

The findings suggest that moving distributed loads represent practical loading conditions more accurately than idealized point loads. Consequently, engineers should incorporate realistic load lengths during dynamic bridge analysis to avoid overestimating structural vibration.

Figure 4.0 Effect of Beam Flexural Rigidity

Beam flexural rigidity, represented by the product (EI), is one of the most influential structural parameters governing dynamic behaviour. The results clearly demonstrate that increasing flexural rigidity substantially reduces beam vibration and Dynamic Amplification Factor.

Higher flexural rigidity enables the beam to resist bending deformation more effectively. Since beam deflection is inversely proportional to flexural rigidity, increasing (EI) directly decreases both static and dynamic displacements. In addition, stiffer beams possess higher natural frequencies, making resonance less likely under ordinary moving load velocities.

The numerical results also show that increases in beam stiffness produce progressively smaller reductions in DAF beyond a certain rigidity level. This indicates that after achieving adequate structural stiffness, further increases in beam rigidity provide diminishing improvements in vibration performance while considerably increasing construction costs.

These observations suggest that selecting an optimum beam stiffness rather than simply maximizing rigidity provides the most economical structural design. Similar conclusions have been reported by Bathe (1996), Clough and Penzien (1993), and Esmailzadeh and Ghorashi (1995).

Figure 5.0 Effect of Variable Foundation Profile

Unlike conventional Winkler foundations with constant stiffness, the present study considered spatially varying elastic foundations in which the support stiffness changes continuously along the beam length.

The results indicate that appropriately varying the foundation stiffness significantly improves dynamic performance. When the foundation stiffness gradually increases toward the beam center, the maximum Dynamic Amplification Factor decreases noticeably compared with the corresponding uniform foundation.

This behavior occurs because the largest bending moments and maximum deflections generally develop near mid-span for simply supported beams. Providing additional support stiffness in this critical region increases the local resistance to deformation precisely where it is most needed. Consequently, vibration energy is dissipated more efficiently, resulting in lower oscillation amplitudes.

The numerical results therefore demonstrate that strategically varying foundation stiffness offers better vibration control than applying the same stiffness uniformly along the entire beam length. Such optimized support systems may reduce material consumption while maintaining excellent structural performance.

This finding is particularly applicable to railway track systems, bridge bearings, buried pipelines, and pavement structures where foundation properties naturally vary because of soil conditions.

Figure 6.0 Effect of Beam Slenderness Ratio

The beam slenderness ratio also exhibits a significant influence on dynamic amplification. Slender beams possess relatively small cross-sectional dimensions compared with their span lengths and therefore exhibit lower flexural stiffness.

The numerical results indicate that increasing beam slenderness causes both beam deflection and Dynamic Amplification Factor to increase steadily. The reduction in bending stiffness allows larger deformations under identical moving loads, thereby increasing vibration amplitudes.

More importantly, slender beams possess lower natural frequencies. As a result, resonance can occur at relatively low moving load velocities that frequently arise in engineering practice. This makes slender beams considerably more susceptible to excessive vibration than stockier structural members.

The findings therefore recommend limiting beam slenderness whenever severe moving dynamic loads are anticipated. Alternatively, designers may compensate for slender geometries by increasing foundation stiffness or selecting materials with higher elastic modulus.

Figure 7.0 Combined Influence of Moving Velocity and Foundation Stiffness

The combined parametric investigation demonstrates that moving load velocity and foundation stiffness interact strongly in determining the beam response. For beams resting on relatively soft foundations, increasing velocity produces a rapid increase in Dynamic Amplification Factor because the beam remains highly flexible. The resonance peak is sharp and occurs over a relatively narrow velocity range.

As the foundation stiffness increases, two important changes become evident. First, the maximum DAF decreases substantially because the beam experiences smaller vibration amplitudes. Second, the resonance peak becomes broader and less pronounced, indicating that the structural response becomes less sensitive to variations in moving load speed.

This interaction illustrates that foundation improvement not only reduces vibration magnitude but also enhances the overall stability of the structural system by reducing the likelihood of severe resonance.

The results therefore suggest that foundation design should be considered simultaneously with vehicle operating speed during the design of transportation structures.

Overall Discussion of the Parametric Investigation

The comprehensive numerical investigation confirms that the Dynamic Amplification Factor of Euler–Bernoulli beams supported by variable elastic foundations is governed by the interaction among moving load velocity, beam flexural rigidity, distributed load length, beam slenderness, and foundation stiffness. Among all the parameters considered, the moving load velocity exerts the greatest influence because it directly controls the dynamic excitation frequency and determines whether resonance conditions develop. Increasing the elastic foundation stiffness and beam flexural rigidity consistently suppresses beam vibrations by increasing the effective stiffness and natural frequencies of the system, thereby reducing both dynamic deflections and the DAF. Conversely, increasing beam slenderness significantly amplifies the response because of reduced bending stiffness and lower natural frequencies. The study further demonstrates that longer distributed loads generate smoother structural responses than shorter loads, while spatially varying foundation stiffness provides more efficient vibration control than uniform foundations by concentrating support where bending demands are greatest. These findings are in close agreement with established moving-load theories and previous studies by Fryba (1999), Hetényi (1946), Esmailzadeh and Ghorashi (1995), Yang et al. (2004), Chen (2003), and Abidi and Khodja (2010). From an engineering standpoint, the results highlight that careful optimization of beam stiffness, foundation characteristics, and allowable operating speeds can substantially reduce dynamic amplification, improve structural serviceability, minimize fatigue damage, and extend the operational lifespan of bridges, railway tracks, elevated guideways, and other beam-supported infrastructure subjected to moving distributed loads.

Conclusion

This study presented a comprehensive parametric investigation of the Dynamic Amplification Factor (DAF) of an elastically supported Euler–Bernoulli beam resting on a variable elastic foundation and subjected to moving distributed loads. An analytical and numerical model was developed to examine the influence of moving load velocity, foundation stiffness, beam flexural rigidity, distributed load length, foundation stiffness variation, and beam slenderness on the dynamic response of the beam–foundation system. The results demonstrated that the moving load velocity is the most dominant parameter governing the Dynamic Amplification Factor. The DAF increased steadily with increasing load velocity and attained its maximum value near the critical velocity due to resonance-like behaviour, after which it decreased as the interaction time between the moving load and the beam became shorter. This finding highlights the necessity of avoiding operating speeds that coincide with the natural frequencies of beam-supported structures.The investigation further revealed that increasing the elastic foundation stiffness significantly improved the dynamic performance of the beam by providing greater restoring forces, reducing beam deflections, increasing the effective structural stiffness, and shifting the natural frequencies to higher values. Similarly, variable elastic foundations, in which the stiffness is strategically increased in regions of maximum bending, provided superior vibration control compared with uniform foundations, demonstrating the effectiveness of optimizing the spatial distribution of foundation stiffness. Beam flexural rigidity was also found to play a critical role in reducing vibration amplitudes and Dynamic Amplification Factors by increasing the resistance of the beam to bending deformation, although the improvement became less significant beyond an optimum stiffness level. The study also established that the characteristics of the moving distributed load strongly influence the structural response. Longer distributed loads generated lower Dynamic Amplification Factors than shorter loads because the applied force was distributed more uniformly over the beam span, thereby reducing localized bending effects and dynamic excitation. Conversely, increasing the beam slenderness ratio resulted in larger beam deflections and higher Dynamic Amplification Factors due to reduced flexural stiffness and lower natural frequencies, indicating that slender beams are more susceptible to excessive dynamic vibrations. The combined parametric analysis further demonstrated that increasing the foundation stiffness not only reduced the magnitude of the Dynamic Amplification Factor but also flattened the resonance peak, making the beam response less sensitive to variations in moving load speed and enhancing overall dynamic stability. Overall, the study confirms that the dynamic behavior of Euler–Bernoulli beams on variable elastic foundations is governed by the complex interaction between structural properties, foundation characteristics, and moving load parameters. The analytical and numerical results show that appropriate optimization of beam stiffness, foundation stiffness distribution, and operating load speeds can effectively suppress resonance, reduce dynamic amplification, improve serviceability, minimize fatigue damage, and extend the service life of engineering structures. The developed model therefore provides a reliable tool for the analysis and design of bridges, railway tracks, highway pavements, airport runways, conveyor systems, crane girders, and other beam-supported infrastructure subjected to moving distributed loads. The findings contribute to the advancement of structural dynamics by demonstrating that variable elastic foundation design offers a practical and economical approach to vibration control, enabling engineers to achieve improved structural safety, durability, and long-term performance under dynamic loading conditions.

References

Abidi, M. and Khodja, M. (2010) 'Dynamic response of Euler–Bernoulli beams resting on elastic foundations under moving loads', *Journal of Sound and Vibration*, 329(11), pp. 2248–2262.

Akin, J.E. (2005) *Finite Element Analysis with Error Estimators: An Introduction to the FEM and Adaptive Error Analysis for Engineering Students*. Oxford: Elsevier.

Bathe, K.J. (1996) *Finite Element Procedures*. Upper Saddle River, NJ: Prentice Hall.

Chen, Y.H. (2003) 'Dynamic stability of beams under moving loads', *International Journal of Solids and Structures*, 40(4), pp. 1057–1072. https://doi.org/10.1016/S0020-7683(02)00648-6.

Chopra, A.K. (2020) *Dynamics of Structures: Theory and Applications to Earthquake Engineering*. 5th edn. Harlow: Pearson.

Clough, R.W. and Penzien, J. (1993) *Dynamics of Structures*. 2nd edn. New York: McGraw–Hill.

Cook, R.D., Malkus, D.S., Plesha, M.E. and Witt, R.J. (2002) *Concepts and Applications of Finite Element Analysis*. 4th edn. New York: John Wiley & Sons.

Esmailzadeh, E. and Ghorashi, M. (1995) 'Vibration analysis of beams subjected to moving loads', *Journal of Sound and Vibration*, 184(1), pp. 9–17.

Frýba, L. (1999) *Vibration of Solids and Structures under Moving Loads*. 3rd edn. London: Thomas Telford Publishing.

Gere, J.M. and Goodno, B.J. (2013) *Mechanics of Materials*. 8th edn. Stamford, CT: Cengage Learning.

Hetényi, M. (1946) *Beams on Elastic Foundation*. Ann Arbor: University of Michigan Press.

Humar, J.L. (2012) *Dynamics of Structures*. 3rd edn. Boca Raton, FL: CRC Press.

Idowu, I. A., Atilade, A. O., Okedeyi, A. S. and Mustapha, R. A. (2020). Dynamic response of Euler-Bernoulli beam on variable pre-stressed under a partially distributed moving load. Nigerian Journal of Physics, 29(2), 195–200.

Idowu, I. A., Atilade, A. O. and Okedeyi, A. S. (2021). Analysis of Bernoulli-Euler beam on variable K-stiffness subjected to partially distributed moving load. Nigerian Journal of Physics, 30(2), 114–120.

Li, J. and Su, M. (2018) 'Dynamic analysis of beams on non-uniform elastic foundations subjected to moving distributed loads', *Engineering Structures*, 165, pp. 258–270.

Meirovitch, L. (2001) *Fundamentals of Vibrations*. New York: McGraw–Hill.

Ouyang, H. (2011) 'Moving-load dynamic problems: A tutorial (with a brief overview)', *Mechanical Systems and Signal Processing*, 25(6), pp. 2039–2060. https://doi.org/10.1016/j.ymssp.2010.12.010.

Rao, S.S. (2017) *Mechanical Vibrations*. 6th edn. Harlow: Pearson.

Reddy, J.N. (2019) *Energy Principles and Variational Methods in Applied Mechanics*. 3rd edn. Hoboken, NJ: John Wiley & Sons.

Timoshenko, S.P. and Gere, J.M. (1972) *Mechanics of Materials*. New York: Van Nostrand Reinhold.

Timoshenko, S.P. and Young, D.H. (1968) *Vibration Problems in Engineering*. 4th edn. New York: John Wiley & Sons.

Weaver, W., Timoshenko, S.P. and Young, D.H. (1990) *Vibration Problems in Engineering*. 5th edn. New York: John Wiley & Sons.

Wu, J.J. (2001) 'Vibration analyses of beams subjected to moving distributed masses using finite element methods', *Journal of Sound and Vibration*, 239(4), pp. 747–764.

Xu, Y.L., Xia, H. and Zhang, N. (2006) 'Dynamic analysis of coupled vehicle–bridge systems under moving loads', *Journal of Sound and Vibration*, 292(3–5), pp. 1036–1059.

Yang, Y.B., Yau, J.D. and Wu, Y.S. (2004) *Vehicle–Bridge Interaction Dynamics: With Applications to High-Speed Railways*. Singapore: World Scientific Publishing.
Zhai, W.M., Wang, K.Y. and Cai, C.B. (2009) 'Fundamentals of vehicle–track coupled dynamics', *Vehicle System Dynamics*, 47(11), pp. 1349–1376. https://doi.org/10.1080/00423110802621561.
Zienkiewicz, O.C., Taylor, R.L. and Zhu, J.Z. (2013) *The Finite Element Method: Its Basis and Fundamentals*. 7th edn. Oxford: Elsevier.